Existence results for the flow of viscoelastic fluids with an integral constitutive law

Laurent Chupin

Abstract. We consider the flows of viscoelastic fluid which obey a constitutive law of integral type. Some theoretical results are proved: local existence, global existence with small data and uniqueness results for the initial boundary value problem.

1. Introduction

The objective of this paper is to provide mathematical results on the integral models for viscoelastic flows. In particular, we are interested in the existence and the uniqueness of a strong solution.

In the integral models, stress components $\boldsymbol{\tau}$ are obtained by integrating appropriate functions, representing the amount of deformation, and taking into account all the strain history of the fluid. Precisely, the constitutive law given the stress at time t and at position $\boldsymbol{x}$ can be written as follows:

$$\boldsymbol{\tau}(t,\boldsymbol{x}) = \int_{-\infty}^{t} m(t-T)\,\mathcal{S}\big(\boldsymbol{F}(T,t,\boldsymbol{x})\big)\,\mathrm{d}T. \tag{1.1}$$

The scalar function m (the memory) and the tensorial function $\mathcal{S}$ depend on the properties of the fluids studied, whereas the deformation tensor $\boldsymbol{F}$ is coupled with the velocity field of the flow. This flow is itself governed by the Navier-Stokes equations, this constitutes a very strong coupling between the velocity and the stress.

In some rare cases, it is possible to express the integral models into differential forms (as in the Maxwell models which corresponds to the case where $\mathcal{S}$ is linear and where the memory m exponentially decreases). For these differential models, there are many mathematical results in the same spirit as those presented here (see for instance [10, 15, 21, 22, 32, 33]).

But for really integral models there are far fewer results. The only relevant work on this type of model is that of M. Renardy [23, 36, 37]. For instance, in [37], M. Renardy proves an existence and uniqueness result for a K-BKZ fluid using Kato's theory of quasilinear hyperbolic equations. Its elegant approach differs substantially from the approach used here, and does not seem easily adaptable to more general laws. In particular the memory function m is not singular at 0 contrary to what is predicted by some molecular models like the Doi-Edwards model. Notice that the case of a singular memory function is then studied in [23] but, like in the previous paper, the results are only local in time. It is important to note that, while the theoretical results are very few, many authors have studied the numerical simulation of flows with an integral law of type (1.1). The review article [28] and references cited therein, provide a good overview of the state of the art regarding the various methods.

The main reason for this lack of theoretical results is probably the nature of the equations:
– To evaluate the stress in an integral model, we must know all the previous configurations. In the present paper, this difficulty is overcome by introducing an additional time variable corresponding to the age. It is then necessary to manage two different times. We prove a Gronwall type lemma in two variables in order to obtain fine estimates with respect to these variables.
– The usual integral models are strongly nonlinear (in the linear case we find the well-known Maxwell models). A possibility to circumvent this difficulty is to work with solutions regular enough. More precisely we strongly used Banach algebras such as the Sobolev spaces $W^{1,p}$ for p large enough, typically for p greater than the dimension of the physical fluid domain.

Organization of the paper – Section 2 is devoted to the presentation of the model. The dimensionless form of equations and many classic examples are given. The main results are stated in Section 3 whereas the proofs are detailed in the following sections. Section 4 is entirely devoted to the proof of the first theorem regarding the local (in time) well posedness. The next two Sections (5 and 6) are devoted to the proof of the uniqueness result and the global existence with small data respectively. The conclusion of this paper (Section 7) contains many remarks and open questions. Finally, some notions on tensors, and a technical Gronwall type lemma have been postponed to Appendices A and B.

2. Governing equations

2.1. Conservation principles

The fluid flows is modeled using the equation of conservation of the linear momentum and the equation of the conservation of mass. In the incompressible and isothermal case these conservation laws read as follows:

$$\begin{cases} \rho\,(\partial_t \boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}) + \nabla p = \operatorname{div}\boldsymbol{\sigma} + \boldsymbol{f}, \\ \operatorname{div}\boldsymbol{u} = 0. \end{cases} \tag{2.1}$$

The real ρ is the constant density of mass and the vector $\boldsymbol{f}$ corresponds to some external body forces. This system is closed using a constitutive equation connecting the stress and the deformation $D\boldsymbol{u} = \frac{1}{2}(\nabla \boldsymbol{u} + {}^T(\nabla \boldsymbol{u}))$.
For a Newtonian viscous fluid, the relationship is linear: $\boldsymbol{\tau}_s = 2\eta_s D\boldsymbol{u}$. The real $\eta_s > 0$ is named the solvent viscosity and the contribution $\operatorname{div} \boldsymbol{\tau}_s$ in the momentum equation gives the usual diffusive term $\eta_s \Delta \boldsymbol{u}$. To taking account some elasticity aspect, which characterise polymer solutions, we add to the viscous contribution $\boldsymbol{\tau}_s$ an elastic one:

$$\boldsymbol{\sigma} = 2\eta_s D\boldsymbol{u} + \boldsymbol{\tau}. \tag{2.2}$$

The role of this additional contribution $\boldsymbol{\tau}$ is to take into account the past history of the fluid. The most natural way to do this is to introduce the integral models.

2.2. Integral models

The elastic contribution $\boldsymbol{\tau}(t, \boldsymbol{x})$ at a time t and at a spatial position $\boldsymbol{x}$ is usually written

$$\boldsymbol{\tau}(t, \boldsymbol{x}) = \underset{T<t}{\mathfrak{F}} (\boldsymbol{F}(T, t, \boldsymbol{x})), \tag{2.3}$$

where $\mathfrak{F}$ is a functional to clarify, which depends on the deformation gradient $\boldsymbol{F}(T, t, \cdot)$ from a times T to a future time t. This approach is classical and widely used in the field of continuum mechanics, see for instance the recent review [38] by J.-C. Saut.
More precisely, the deformation gradient $\boldsymbol{F}(T, t, \cdot)$ measures stretch and rotation. It is defined as follows: for two times $T \leq t$ given, we first introduce the notation $\boldsymbol{x}(T, t, \boldsymbol{X})$ which corresponds to the position at time t of the fluid particle which was at the position $\boldsymbol{X}$ at time T. The dynamics of any mechanical problem with a velocity field $\boldsymbol{u}(t, \boldsymbol{x})$ can be described by this flow map $\boldsymbol{x}(T, t, \boldsymbol{X})$ which is a time dependent family of orientation preserving diffeomorphisms:

$$\begin{cases} \partial_t \boldsymbol{x}(T, t, \boldsymbol{X}) = \boldsymbol{u}(t, \boldsymbol{x}(T, t, \boldsymbol{X})), \\ \boldsymbol{x}(T, T, \boldsymbol{X}) = \boldsymbol{X}. \end{cases} \tag{2.4}$$

The deformation gradient $\widetilde{\boldsymbol{F}}(T, t, \boldsymbol{X})$ is used to describe the changing of any configuration, amplification or pattern during the dynamical process. It is defined by

$$\widetilde{\boldsymbol{F}}(T, t, \boldsymbol{X}) = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}}(T, t, \boldsymbol{X}). \tag{2.5}$$

The deformation gradient $\boldsymbol{F}(T, t, \boldsymbol{x})$ will be finally defined as the corresponding in the Eulerian coordinates:

$$\boldsymbol{F}(T, t, \boldsymbol{x}(T, t, \boldsymbol{X})) = \widetilde{\boldsymbol{F}}(T, t, \boldsymbol{X}). \tag{2.6}$$

The integral models we study in this article correspond to the particular case of Equation (2.3). They are written

$$\boldsymbol{\tau}(t, \boldsymbol{x}) = \int_{-\infty}^{t} m(t - T)\, \mathcal{S}\big(\boldsymbol{F}(T, t, \boldsymbol{x})\big)\, \mathrm{d}T, \tag{2.7}$$

where m is called a memory function. The function $\mathcal{S}$ is a model-dependent strain measure. It is a tensorial function (its arguments and its images are 2-tensors).

Remark 2.1. Due to many physical principles, the functions m and $\mathcal{S}$ must satisfy some assumptions.
– For instance, the principle of frame indifference implies that the stress tensor depends on the relative deformation gradient $\boldsymbol{F}$ only through the relative Finger tensor ${}^T\boldsymbol{F} \cdot \boldsymbol{F}$ (or its inverse, the Cauchy-Green tensor), see the examples given in Subsection 2.4.
– In the same way (see also Subsection 2.4), the principle of fading memory implies that m must be a positive function which decreases to 0.

2.3. A closed system

2.3.1. PDE for the deformation gradient. From the definition (2.4)–(2.5) of the deformation gradient, it is possible to deduce a partial differential equation coupling this deformation $\boldsymbol{F}$ and the velocity field $\boldsymbol{u}$, see for instance [24]: We derivate the relation (2.6) with respect to the time t. The chain rule together with the relation (2.4) yields the following equation

$$\begin{aligned} \partial_t \widetilde{\boldsymbol{F}}(T,t,\boldsymbol{X}) &= \partial_t \boldsymbol{F}(T,t,\boldsymbol{x}) + \partial_t \boldsymbol{x}(T,t,\boldsymbol{X}) \cdot \partial_{\boldsymbol{x}} \boldsymbol{F}(T,t,\boldsymbol{x}) \\ &= \partial_t \boldsymbol{F}(T,t,\boldsymbol{x}) + \boldsymbol{u}(t,\boldsymbol{x}) \cdot \partial_{\boldsymbol{x}} \boldsymbol{F}(T,t,\boldsymbol{x}). \end{aligned} \tag{2.8}$$

But using the relation (2.5) together with the chain rule and the relation (2.4) again, we obtain

$$\begin{aligned} \partial_t \widetilde{\boldsymbol{F}}(T,t,\boldsymbol{X}) &= \partial_{\boldsymbol{X}} (\partial_t \boldsymbol{x}(T,t,\boldsymbol{X})) \\ &= \partial_{\boldsymbol{X}} (\boldsymbol{u}(t,\boldsymbol{x}(T,t,\boldsymbol{X}))) \\ &= \partial_{\boldsymbol{X}} \boldsymbol{x}(T,t,\boldsymbol{X}) \cdot \partial_{\boldsymbol{x}} \boldsymbol{u}(t,\boldsymbol{x}) \\ &= \boldsymbol{F}(T,t,\boldsymbol{x}) \cdot \partial_{\boldsymbol{x}} \boldsymbol{u}(t,\boldsymbol{x}). \end{aligned} \tag{2.9}$$

Equations (2.8) and (2.9) show that we have the following relation coupling the velocity field $\boldsymbol{u}$ and the deformation gradient $\boldsymbol{F}$:

$$\partial_t \boldsymbol{F} + \boldsymbol{u} \cdot \nabla \boldsymbol{F} = \boldsymbol{F} \cdot \nabla \boldsymbol{u}. \tag{2.10}$$

2.3.2. A new time variable to take into account the past. Note that in the previous subsection, the time T can be view as a parameter. In fact, it is only used in the law (2.3), or in the law (2.7) for the integral form, as a marker of past events. In the sequel, it is interesting to select as independent variable the age $s = t - T$, which is measured relative to the current time t. This viewpoint is relatively classical in the numerical framework, see for instance [25, 28, 43]. Now we introduce $\boldsymbol{G}(s,t,\boldsymbol{x}) = \boldsymbol{F}(t-s,t,\boldsymbol{x})$. Clearly, we have the following relation instead of the relation (2.10):

$$\partial_t \boldsymbol{G} + \partial_s \boldsymbol{G} + \boldsymbol{u} \cdot \nabla \boldsymbol{G} = \boldsymbol{G} \cdot \nabla \boldsymbol{u}, \tag{2.11}$$

where naturally the velocity $\boldsymbol{u}$ only depends on $(t,\boldsymbol{x})$ and is independent of this new variable s. Moreover, in term of variables (s,t), the relation (2.7) given the stress tensor reads

$$\boldsymbol{\tau}(t,\boldsymbol{x}) = \int_0^{+\infty} m(s)\, \mathcal{S}\big(\boldsymbol{G}(s,t,\boldsymbol{x})\big)\, \mathrm{d}s. \tag{2.12}$$

Initial and past conditions – In order to describe a flow, we must at least know it in its original configuration. For a memory flow, we need to know all the past of the flow. More precisely, if we denote by $t = 0$ this initial time that we consider, then we assume that there exists a given function $\boldsymbol{F}_{\mathrm{old}}$ such that

$$\boldsymbol{F}(T,0,\boldsymbol{x}) = \boldsymbol{F}_{\mathrm{old}}(T,\boldsymbol{x}) \qquad \forall T \leq 0. \tag{2.13}$$

For $T > 0$ fixed, the deformation field $\boldsymbol{F}(T,t,\boldsymbol{x})$ can be thought of as having been created at time $t = T$ with the natural initial condition

$$\boldsymbol{F}(T,T,\boldsymbol{x}) = \boldsymbol{\delta} \qquad \forall T \geq 0, \tag{2.14}$$

the symbol $\boldsymbol{\delta}$ corresponding to the identity tensor represented by the matrix $\boldsymbol{\delta}_{ij} = 1$ if $i = j$ and $\boldsymbol{\delta}_{ij} = 0$ otherwise.

Remark 2.2. The relations (2.13) and (2.14) must clearly be compatible: we must have the equality $\boldsymbol{F}_{\mathrm{old}}(0,\boldsymbol{x}) = \boldsymbol{\delta}$. In many articles, the fluid is assumed to be quiescent before the initial time so that the authors use $\boldsymbol{F}_{\mathrm{old}}(T,\boldsymbol{x}) = \boldsymbol{\delta}$ for all $T \leq 0$, see for instance [25, 28, 43].

In term of the new variables (s,t) the two relations (2.13) and (2.14) read

$$\begin{aligned} \boldsymbol{G}(s,0,\boldsymbol{x}) &= \boldsymbol{G}_{\mathrm{old}}(s,\boldsymbol{x}) && \forall s \geq 0,\\ \boldsymbol{G}(0,t,\boldsymbol{x}) &= \boldsymbol{\delta} && \forall t \geq 0, \end{aligned}$$

where $\boldsymbol{G}_{\mathrm{old}}(s,\boldsymbol{x}) = \boldsymbol{F}_{\mathrm{old}}(-s,\boldsymbol{x})$.

2.3.3. Non-dimensional final model. The resulting system using Equations (2.1), (2.2), (2.11) and (2.12) is then written

$$\begin{cases} \rho\left(\partial_t \boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}\right) + \nabla p - \eta_s \Delta\boldsymbol{u} = \operatorname{div}\boldsymbol{\tau} + \boldsymbol{f},\\ \operatorname{div}\boldsymbol{u} = 0,\\ \boldsymbol{\tau}(t,\boldsymbol{x}) = \displaystyle\int_0^{+\infty} m(s)\, \mathcal{S}\big(\boldsymbol{G}(s,t,\boldsymbol{x})\big)\, \mathrm{d}s,\\ \partial_t \boldsymbol{G} + \partial_s \boldsymbol{G} + \boldsymbol{u}\cdot\nabla\boldsymbol{G} = \boldsymbol{G}\cdot\nabla\boldsymbol{u}. \end{cases} \tag{2.15}$$

This system is written in a non-dimensional form in the usual way. We introduce the characteristic values U and L for the velocity and the length. The current time is then of order of L/U and it is natural to introduce another characteristic time λ for the age variable s. The characteristic viscosity of the fluid takes into account the viscosity η_s of the solvent, but also the viscosity η_e of the elastic part (the polymer): we note $\eta = \eta_s + \eta_e$. More precisely, we introduce the following dimensionless variables, quoted by a star:

$$\boldsymbol{x}^\star = \frac{\boldsymbol{x}}{L}, \qquad \boldsymbol{u}^\star = \frac{\boldsymbol{u}}{U}, \qquad t^\star = \frac{t}{L/U}, \qquad s^\star = \frac{s}{\lambda},$$
$$p^\star = \frac{p}{\eta U/L}, \qquad \boldsymbol{\tau}^\star = \frac{\boldsymbol{\tau}}{\eta U/L}, \qquad \boldsymbol{f}^\star = \frac{\boldsymbol{f}}{\eta U/L^2},$$
$$\boldsymbol{G}^\star = \boldsymbol{G}, \qquad \mathcal{S}^\star(\boldsymbol{G}^\star) = \frac{\mathcal{S}(\boldsymbol{G})}{\eta_e/\lambda}, \qquad m^\star(s^\star) = \frac{m(s)}{1/\lambda}.$$

System (2.15) is then written in dimensionless form as follows (where we drop the star for sake of simplicity):

$$\begin{cases} \mathfrak{Re}(\partial_t \boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}) + \nabla p - (1-\omega)\Delta\boldsymbol{u} = \operatorname{div}\boldsymbol{\tau} + \boldsymbol{f}, \\ \operatorname{div}\boldsymbol{u} = 0, \\ \boldsymbol{\tau}(t,\boldsymbol{x}) = \dfrac{\omega}{\mathfrak{We}} \displaystyle\int_0^{+\infty} m(s)\, \mathcal{S}\big(\boldsymbol{G}(s,t,\boldsymbol{x})\big)\, \mathrm{d}s, \\ \partial_t \boldsymbol{G} + \dfrac{1}{\mathfrak{We}}\partial_s \boldsymbol{G} + \boldsymbol{u}\cdot\nabla\boldsymbol{G} = \boldsymbol{G}\cdot\nabla\boldsymbol{u}, \end{cases} \tag{2.16}$$

where we introduced the three non-dimensional numbers which characterize the flow:

- The Reynolds number $\mathfrak{Re} = \frac{\rho U L}{\eta}$ which corresponds to the ratio between inertial and viscous forces acting on the fluid;
- The Weissenberg number $\mathfrak{We} = \frac{\lambda U}{L}$ which is the ratio between the time of the relaxation of the fluid and the time of the experiment;
- The retardation parameter $\omega = \frac{\eta_e}{\eta} \in [0,1]$ which balances the purely viscous effects ($\omega = 0$) and the purely elastic effects ($\omega = 1$).

System (2.16) is closed with the following initial and boundary conditions:

$$\boldsymbol{u}\big|_{t=0} = \boldsymbol{u}_0, \qquad \boldsymbol{u}\big|_{\partial\Omega} = \boldsymbol{0}, \qquad \boldsymbol{G}\big|_{t=0} = \boldsymbol{G}_{\mathrm{old}}, \qquad \boldsymbol{G}\big|_{s=0} = \boldsymbol{\delta}. \tag{2.17}$$

Remark 2.3. As we said in Remark 2.1, the stress tensor $\boldsymbol{\tau}$ depends on the deformation tensor $\boldsymbol{G}$ *via* the right relative Finger tensor $\boldsymbol{B} = {}^T\boldsymbol{G}\cdot\boldsymbol{G}$ or *via* the Green-Cauchy tensor $\boldsymbol{C} = \boldsymbol{B}^{-1}$. Using the last equation of (2.16), we note that the tensor $\boldsymbol{B}$ satisfies

$$\partial_t \boldsymbol{B} + \frac{1}{\mathfrak{We}}\partial_s \boldsymbol{B} + \boldsymbol{u}\cdot\nabla\boldsymbol{B} = \boldsymbol{B}\cdot\nabla\boldsymbol{u} + {}^T(\nabla\boldsymbol{u})\cdot\boldsymbol{B}, \tag{2.18}$$

whereas the tensor $\boldsymbol{C}$ satisfies

$$\partial_t \boldsymbol{C} + \frac{1}{\mathfrak{We}}\partial_s \boldsymbol{C} + \boldsymbol{u}\cdot\nabla\boldsymbol{C} = -\boldsymbol{C}\cdot{}^T(\nabla\boldsymbol{u}) - \nabla\boldsymbol{u}\cdot\boldsymbol{C}. \tag{2.19}$$

2.4. Examples of integral models

In this section we present some classical integral laws of kind (2.12) to model the viscoelasticity. These law are defined by the memory function m and by the strain measure $\mathcal{S}$.

2.4.1. Memory function m. In accordance with thermodynamics through what is called the principle of fading memory - see [11] - the memory function m used in the integral models (2.12) must be decreasing, positive and must satisfy $\int_0^\infty m(s)\,\mathrm{d}s = 1$.
In many cases experimentally observed relaxation functions exhibit a stretched exponential decay $\mathrm{e}^{-(s/\lambda)}$ where $\lambda > 0$ is a relaxation time. We could as well have considered the case of several relaxation times, that is a memory function like

$$m(s) = \sum_{k=1}^{N} \frac{\eta_k}{\lambda_k^2} \mathrm{e}^{-s/\lambda_k}. \tag{2.20}$$

From a mathematical point of view, it will be equivalent to consider the memory function $m(s) = \mathrm{e}^{-s}$ (in dimensionless form).
This expression for the memory function can be generalized. For instance, in the Doi-Edwards model - see [13], the memory function is given by

$$m(s) = \frac{8}{\pi^2\,\lambda} \sum_{k=0}^{+\infty} \mathrm{e}^{-(2k+1)^2 s/\lambda}. \tag{2.21}$$

In practice, the difference between this model (2.21) and the model (2.20) containing a finite number of relaxation times is really important. In fact, in the case of the model (2.21), the function m has a singularity in 0. This singularity can bring additional difficulties (eg, such a case is not treated in the article [37] of M. Renardy, and treated in [23] with other assumptions on the memory such that the integrability of the derivatives m' and m''). The function m remains integrable, which is the key assumption for the present results.

Even if the exponential case is usually used, many other possible choices for the memory function m are possibles - see [16]. The algebraic pattern $g(s) = (s/\lambda)^{-\beta}$ with $0 < \beta < 1$ is observed in the stress relaxation of viscoelastic materials such as critical gels [8, 40], in the charge carrier transport in amorphous semiconductors [39], in dielectric relaxation [26] or in the attenuation of seismic waves [30]. That corresponds to the following memory functions

$$m(s) = \sum_{k=1}^{N} \frac{\eta_k\,\beta_k}{\lambda_k} \Big(\frac{s}{\lambda_k}\Big)^{-(\beta_k+1)}.$$

2.4.2. Strain measure $\mathcal{S}$. The more simple case corresponds to the choice $\mathcal{S}(\boldsymbol{G}) = \boldsymbol{B} - \boldsymbol{\delta}$ where $\boldsymbol{B} = {}^{T}\boldsymbol{G}\cdot\boldsymbol{G}$, and where the memory (dimensionless) function m is given by $m(s) = \mathrm{e}^{-s}$:
Maxwell models – The stress tensor $\boldsymbol{\tau}$ is then given by

$$\boldsymbol{\tau}(t,\boldsymbol{x}) = \frac{\omega}{\mathfrak{We}} \int_0^{+\infty} \mathrm{e}^{-s}\big(\boldsymbol{B}(s,t,\boldsymbol{x}) - \boldsymbol{\delta}\big)\,\mathrm{d}s.$$

This expression is simple enough to deduce a PDE for the stress tensor $\boldsymbol{\tau}$ from the PDE for the deformation tensor $\boldsymbol{G}$. In fact we use Equation (2.18) satisfied by the Finger tensor:

$$\partial_t(\boldsymbol{B}-\boldsymbol{\delta})+\frac{1}{\mathfrak{We}}\partial_s(\boldsymbol{B}-\boldsymbol{\delta})+\boldsymbol{u}\cdot\nabla(\boldsymbol{B}-\boldsymbol{\delta}) = (\boldsymbol{B}-\boldsymbol{\delta})\cdot\nabla\boldsymbol{u}+{}^T(\nabla\boldsymbol{u})\cdot(\boldsymbol{B}-\boldsymbol{\delta})+2D\boldsymbol{u}.$$

We next multiply this equation by $\frac{\omega}{\mathfrak{We}}m(s)$ and integrate for $s\in(0,+\infty)$. Taking into account the initial condition $\boldsymbol{B}|_{s=0}=\boldsymbol{\delta}$, we obtain the Upper Convected Maxwell (UCM) model:

$$\mathfrak{We}\Big(\partial_t\boldsymbol{\tau}+\boldsymbol{u}\cdot\nabla\boldsymbol{\tau}-{}^T(\nabla\boldsymbol{u})\cdot\boldsymbol{\tau}-\boldsymbol{\tau}\cdot\nabla\boldsymbol{u}\Big)+\boldsymbol{\tau}=2\omega D\boldsymbol{u}.$$

Another classical case is to the Lower Convected Maxwell (LCM) model. It corresponds to $\mathcal{S}(\boldsymbol{G})=\boldsymbol{\delta}-\boldsymbol{C}$ where $\boldsymbol{C}=\boldsymbol{B}^{-1}$, and to an exponential memory function $m(s)=\mathrm{e}^{-s}$. Using Equation (2.19) we obtain, like to get the UCM model:

$$\mathfrak{We}\Big(\partial_t\boldsymbol{\tau}+\boldsymbol{u}\cdot\nabla\boldsymbol{\tau}+\boldsymbol{\tau}\cdot{}^T(\nabla\boldsymbol{u})+\nabla\boldsymbol{u}\cdot\boldsymbol{\tau}\Big)+\boldsymbol{\tau}=2\omega D\boldsymbol{u}.$$

Remark 2.4. In fact there exists a continuum of such model (called Oldroyd models and corresponding to a balance between the upper-convected model and the lower-convected one) but we do not know if these models derive from integral models.

K-BKZ models – Among the most relevant nonlinear cases, the most popular integral models for a viscoelastic flow are the K-BKZ models introduce by B. Bernstein, E. A. Kearsley and L. J. Zapas [4, 5] and A. Kaye [27]. For such models, $\mathcal{S}$ takes the following form:

$$\mathcal{S}(\boldsymbol{G})=\phi_1(I_1,I_2)(\boldsymbol{B}-\boldsymbol{\delta})+\phi_2(I_1,I_2)(\boldsymbol{\delta}-\boldsymbol{C}),$$

where ϕ_1 and ϕ_2 are two scalar functions of the strain invariants $I_1=\mathrm{Tr}(\boldsymbol{B})$ and $I_2=\mathrm{Tr}(\boldsymbol{B}^{-1})$ (see Appendix A for a discussion on these invariants). Clearly, the two Maxwell models presented in the previous paragraph are particular K-BKZ models, for $(\phi_1,\phi_2)=(1,0)$ and $(\phi_1,\phi_2)=(0,1)$.

Remark 2.5. In the integral models presented here, if we consider $\mathcal{S}(\boldsymbol{G})+\phi\,\boldsymbol{\delta}$ instead of $\mathcal{S}(\boldsymbol{G})$ then $\boldsymbol{\tau}$ becomes $\boldsymbol{\tau}+\frac{\omega}{\mathfrak{We}}\phi\,\boldsymbol{\delta}$ and the additional contribution can be considered as a pressure contribution in the Navier-Stokes equation on the velocity field. Then such modification has no influence on the mathematical structure of the whole System (2.16).

Following the last remark, we can see the PSM models presented by A. C. Papanastasiou, L. Scriven and C. Macosko in [35] as K-BKZ models:

$$\begin{aligned}&\mathcal{S}(\boldsymbol{G})=h(I_1,I_2)\boldsymbol{B}\\ &\text{with}\quad h(I_1,I_2)=\frac{\alpha}{\alpha+\beta I_1+(1-\beta)I_2-3}.\end{aligned}\tag{2.22}$$

In these models, the parameters $\alpha>0$ and $0\le\beta\le1$ are obtained from the rheological fluid properties. In the same way, Wagner [44, 45] proposes the same law

$$\text{with}\quad h(I_1,I_2)=\exp(-\alpha\sqrt{\beta I_1+(1-\beta)I_2-3}).\tag{2.23}$$

Doi-Edwards model – The Doi-Edwards model is a molecular model where the motion of the polymers is described by reptation in a tube, more precisely it corresponds to the simplest tube model of entangled linear polymers. The memory function m associated to such model is given by the relation (2.21) whereas the strain measure is obtained as an average with respect to the orientation of tube segments. Works of P.-K. Currie show that we can approach this model using the following strain function (named the Currie approximation, see [12]):

$$\mathcal{S}(\boldsymbol{G}) = \frac{4}{3(J-1)}\boldsymbol{B} - \frac{4}{3(J-1)\sqrt{I_2+3.25}}\boldsymbol{C} \qquad \text{where} \quad J = I_1 + 2\sqrt{I_2+3.25}. \tag{2.24}$$

3. Main results

3.1. Mathematical framework

In the sequel, the fluid domain $\Omega \subset \mathbb{R}^d$, $d \geq 2$, is a bounded connected open set with a smooth boundary $\partial\Omega$ (in fact the regularity $\mathcal{C}^{2,\alpha}$ for some $\alpha > 0$ is sufficient as in the proof dedicated to the Oldroyd case, see [14, 15]). We use the following standard notations:

– For all real $s \geq 0$ and all integer $p \geq 1$, the set $W^{s,p}(\Omega)$ corresponds to the usual Sobolev spaces. We classically note $L^p(\Omega) = W^{0,p}(\Omega)$ and $H^s(\Omega) = W^{s,2}(\Omega)$.

We will frequently use functions with values in $\mathbb{R}^d$ or in the space $\mathcal{L}(\mathbb{R}^d)$ of real $d \times d$ matrices. In all cases, the notations will be abbreviated. For instance, the space $(W^{1,p}(\Omega))^3$ will be denoted $W^{1,p}(\Omega)$. Moreover, all the norms will be denoted by index, for instance like $\|\boldsymbol{u}\|_{W^{1,p}(\Omega)}$.

– Since we are interested in the incompressible flows, we introduce

$$H_p(\Omega) = \{\boldsymbol{v} \in L^p(\Omega)\,;\ \operatorname{div}\boldsymbol{v} = 0,\ \boldsymbol{v}\cdot\boldsymbol{n} = 0 \text{ on } \partial\Omega\},$$

where $\boldsymbol{n}$ is the unitary vector normal to $\partial\Omega$, oriented towards the exterior of Ω. Moreover, we note $V(\Omega) = H_2(\Omega) \cap H_0^1(\Omega)$ and $V'(\Omega)$ its dual.

– The Stokes operator A_p in $H_p(\Omega)$ is introduced, with domain

$$D(A_p)(\Omega) = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \cap H_p(\Omega),$$

whereas we note

$$D_p^r(\Omega) = \{\boldsymbol{v} \in H_p\,;\ \|\boldsymbol{v}\|_{L^p(\Omega)} + \Big(\int_0^{+\infty} \|A_p \mathrm{e}^{-tA_p}\boldsymbol{v}\|_{L^p(\Omega)}^r \,\mathrm{d}t\Big)^{1/r} < +\infty\}.$$

– The notation of kind $L^r(0,\mathcal{T}; D(A_p))$ denotes the space of r-integrable functions on $(0,\mathcal{T})$, $\mathcal{T} > 0$, with values in $D(A_p)$. Similarly, expressions like $g \in L^\infty(\mathbb{R}^+; L^r(0,\mathcal{T}; L^p(\Omega)))$ means that

$$\sup_{s\in\mathbb{R}^+} \Big(\int_0^{\mathcal{T}} \|g(s,t,\cdot)\|_{L^p(\Omega)}^r \,\mathrm{d}t\Big)^{\frac{1}{r}} < +\infty.$$

Assumptions – About the model (2.16) itself, it uses the two given functions m and $\mathcal{S}$. From a mathematical point of view we assume very general assumptions (satisfied by all the physical models introduced earlier):

(H1) $m : s \in \mathbb{R}^+ \longmapsto m(s) \in \mathbb{R}$ is measurable, positive and $\int_0^{+\infty} m(s)\,\mathrm{d}s = 1$;
(H2) $\mathcal{S} : \boldsymbol{G} \in \mathcal{L}(\mathbb{R}^d) \longmapsto \mathcal{S}(\boldsymbol{G}) \in \mathcal{L}(\mathbb{R}^d)$ is of class $\mathcal{C}^1$.

Note that the notion of derivative for the 2-tensorial application $\mathcal{S}$ will be specified in Appendix A. Moreover, if we wanted to be more accurate, Assumption (H2) is written rather "the function $\mathcal{S}$ is of class $\mathcal{C}^1$ on a subset of $\mathcal{L}(\mathbb{R}^d)$ taking into account the fact that $\det \boldsymbol{G} = 1$" - see Appendix A again. Throughout the remainder of this paper, these two hypotheses (H1) and (H2) will be assumed satisfied.

3.2. Statements of main results

The first result concerns an existence result for strong solutions. It is obviously local with respect to time (as for the results on the Navier-Stokes equations):

Theorem 3.1 (local existence). *Let $\mathcal{T} > 0$, $r \in]1, +\infty[$ and $p \in]d, +\infty[$.*
If $\boldsymbol{u}_0 \in D^r_p(\Omega)$, $\boldsymbol{G}_{\rm old} \in L^\infty(\mathbb{R}^+; W^{1,p}(\Omega))$, $\partial_s \boldsymbol{G}_{\rm old} \in L^r(\mathbb{R}^+; L^p(\Omega))$ and $\boldsymbol{f} \in L^r(0, \mathcal{T}; L^p(\Omega))$ then there exists $\mathcal{T}_\star \in]0, \mathcal{T}]$ and a strong solution $(\boldsymbol{u}, p, \boldsymbol{\tau}, \boldsymbol{G})$ to System (2.16) *in* $[0, \mathcal{T}_\star]$, *which satisfies the initial/boundary conditions* (2.17). *Moreover we have*

$$\begin{array}{ll}
\boldsymbol{u} \in L^r(0, \mathcal{T}_\star; W^{2,p}(\Omega)), & \partial_t \boldsymbol{u} \in L^r(0, \mathcal{T}_\star; L^p(\Omega)), \\
\boldsymbol{\tau} \in L^\infty(0, \mathcal{T}_\star; W^{1,p}(\Omega)), & \partial_t \boldsymbol{\tau} \in L^r(0, \mathcal{T}_\star; L^p(\Omega)), \\
\boldsymbol{G} \in L^\infty(\mathbb{R}^+ \times (0, \mathcal{T}_\star); W^{1,p}(\Omega)), & \partial_s \boldsymbol{G},\ \partial_t \boldsymbol{G} \in L^\infty(\mathbb{R}^+; L^r(0, \mathcal{T}_\star; L^p(\Omega))).
\end{array}$$

We will show that the solution obtained in Theorem 3.1 is the only one in the class of regular solution. Precisely, the result reads as follow

Theorem 3.2 (uniqueness). *Let $\mathcal{T} > 0$.*
If $\boldsymbol{u}_0 \in H$, $\boldsymbol{G}_{\rm old} \in L^\infty(\mathbb{R}^+; L^2(\Omega))$, $\boldsymbol{f} \in L^1(0, \mathcal{T}; V'(\Omega))$ and if (2.16)-(2.17) *possess two weak solutions $(\boldsymbol{u}_1, p_1, \boldsymbol{\tau}_1, \boldsymbol{G}_1)$ and $(\boldsymbol{u}_2, p_2, \boldsymbol{\tau}_2, \boldsymbol{G}_2)$ in the usual sense, with for $i \in \{1, 2\}$,*

$$\boldsymbol{u}_i \in L^\infty(0, \mathcal{T}; L^2(\Omega)) \cap L^2(0, \mathcal{T}; H^1(\Omega)) \cap L^1(0, \mathcal{T}; W^{1,\infty}(\Omega))$$
$$\boldsymbol{G}_i \in L^\infty(\mathbb{R}^+ \times (0, \mathcal{T}); W^{1,d}(\Omega))),$$

then they coincide (p_1 and p_2 coincide up to an additive function only depending on t).

If the data are small, it is possible to show that the unique local solution obtained by Theorems 3.1 and 3.2 is defined for all time (up to an assumption on the relaxation parameter ω):

Theorem 3.3 (global existence with small data). *Let $r \in]1, +\infty[$ and $p \in]d, +\infty[$.*
For each $\mathcal{T} > 0$, with the same assumptions as in Theorem 3.1, *there exists $\omega_\mathcal{T} \in (0, 1)$ such that if $0 \le \omega < \omega_\mathcal{T}$ and if the data $\boldsymbol{u}_0$ and $\boldsymbol{f}$ have sufficiently*

small norms in their respective spaces, then there exists a unique strong solution $(\boldsymbol{u}, p, \boldsymbol{\tau}, \boldsymbol{G})$ *to System* (2.16)-(2.17) *in* $[0, \mathcal{T}]$ *which belongs in the same spaces that the local solution obtained in Theorem* 3.1.

Remark 3.4. We make here some remarks concerning the above three theorems:
- We remark that Theorem 3.3 does not contain smallness conditions on the deformation tensor $\boldsymbol{G}_{\text{old}}$. At rest this tensor is not zero but is equal to the identity tensor. An assumption of smallness should eventually be introduced on the quantity $\boldsymbol{G}_{\text{old}} - \boldsymbol{\delta}$. In fact it is implicit in the smallness assumption on the parameter ω.
- In some papers about the classical Oldroyd model (corresponding to a linear stress relation, see page 7) the smallness condition on the parameter ω is not necessary. It is the case in a Hilbertian framework, that is in the Hilbert spaces H^s instead of the Banach spaces L^p, see [10] or [33].
- Theorem 3.3 can be viewed as a stability result for $\boldsymbol{u} = \mathbf{0}$, $\boldsymbol{\tau} = \mathbf{0}$.

4. Proof of the local existence

This section is devoted to the proof of the local existence theorem 3.1. The main ideas to prove Theorem 3.1 are based on work of C. Guillopé and J.C. Saut [20, 21]. Roughly speaking, we rewrite Equations (2.16) as a fixed point equation and apply Schauder's theorem. This principle was taken up by E. Fernandez-Cara, F. Guillen and R.R. Ortega [14, 15] in the context of the functional spaces $L^r - L^p$. This choice is presented in the present paper. We then analyze three independent problems. A linear Stokes system with given source term and initial value:

$$\begin{cases} \mathfrak{Re}\, \partial_t \boldsymbol{u} + \nabla p - (1-\omega)\Delta \boldsymbol{u} = \overline{\boldsymbol{g}}, \\ \operatorname{div} \boldsymbol{u} = 0, \\ \boldsymbol{u}|_{\partial\Omega} = \mathbf{0}, \quad \boldsymbol{u}|_{t=0} = \boldsymbol{u}_0; \end{cases} \tag{4.1}$$

A problem given the deformation gradient as a function of a given velocity field $\overline{\boldsymbol{u}}$:

$$\begin{cases} \partial_t \boldsymbol{G} + \dfrac{1}{\mathfrak{We}} \partial_s \boldsymbol{G} + \overline{\boldsymbol{u}} \cdot \nabla \boldsymbol{G} = \boldsymbol{G} \cdot \nabla \overline{\boldsymbol{u}}, \\ \boldsymbol{G}|_{s=0} = \boldsymbol{\delta}, \qquad \boldsymbol{G}|_{t=0} = \boldsymbol{G}_{\text{old}}; \end{cases} \tag{4.2}$$

And the constitutive integral law given the stress tensor $\boldsymbol{\tau}$ with respect to a deformation gradient $\overline{\boldsymbol{G}}$:

$$\boldsymbol{\tau}(t, \boldsymbol{x}) = \frac{\omega}{\mathfrak{We}} \int_0^{+\infty} m(s)\, \mathcal{S}\big(\overline{\boldsymbol{G}}(s, t, \boldsymbol{x})\big)\, \mathrm{d}s. \tag{4.3}$$

4.1. Estimates for the velocity $\boldsymbol{u}$ solution of a Stokes problem

The results for the Stokes system (4.1) are very usual. In this subsection we only recall, without proof (we can found a proof in [18]), a well known results for the time dependent Stokes problem:

Lemma 4.1. *Let $\mathcal{T} > 0$, $r \in]1, +\infty[$ and $p \in]1, +\infty[$.*
If $\boldsymbol{u}_0 \in D_p^r(\Omega)$ and $\overline{\boldsymbol{g}} \in L^r(0, \mathcal{T}; H_p)$ then there exists a unique solution $\boldsymbol{u} \in L^r(0, \mathcal{T}; D(A_p))$ such that $\partial_t \boldsymbol{u} \in L^r(0, \mathcal{T}; H_p)$ to Equations (4.1). Moreover this solution satisfies

$$\|\boldsymbol{u}\|_{L^r(0,\mathcal{T};W^{2,p}(\Omega))} + \|\partial_t \boldsymbol{u}\|_{L^r(0,\mathcal{T};L^p(\Omega))} \leq \frac{C_1}{1-\omega}\big(\mathfrak{Re}\, \|\boldsymbol{u}_0\|_{W^{2,p}(\Omega)} + \|\overline{\boldsymbol{g}}\|_{L^r(0,\mathcal{T};L^p(\Omega))}\big),$$

where the constant C_1 only depends on Ω, r and p.

4.2. Estimates for the deformation gradient $\boldsymbol{G}$

This subsection is devoted to the proof of the following lemma, which gives estimates for the solution to System (4.2):

Lemma 4.2. *Let $\mathcal{T} > 0$, $r \in]1, +\infty[$ and $p \in]d, +\infty[$.*
If $\boldsymbol{G}_{\text{old}} \in L^\infty(\mathbb{R}^+; W^{1,p}(\Omega))$, $\partial_s \boldsymbol{G}_{\text{old}} \in L^r(\mathbb{R}^+; L^p(\Omega))$ and $\overline{\boldsymbol{u}} \in L^r(0, \mathcal{T}; D(A_p))$ then the problem (4.2) admits a unique solution $\boldsymbol{G} \in L^\infty(\mathbb{R}^+ \times (0, \mathcal{T}); W^{1,p}(\Omega))$ such that $\partial_s \boldsymbol{G}, \partial_t \boldsymbol{G} \in L^\infty(\mathbb{R}^+; L^r(0, \mathcal{T}; L^p(\Omega)))$. Moreover, this solution satisfies

$$\begin{aligned} &\|\boldsymbol{G}\|_{L^\infty(\mathbb{R}^+\times(0,\mathcal{T});W^{1,p}(\Omega))} \\ &\qquad + \|\partial_s \boldsymbol{G}\|_{L^\infty(\mathbb{R}^+;L^r(0,\mathcal{T};L^p(\Omega)))} \\ &\qquad\qquad + \|\partial_t \boldsymbol{G}\|_{L^\infty(\mathbb{R}^+;L^r(0,\mathcal{T};L^p(\Omega)))} \\ &\quad \leq C_2\big(1 + \|\nabla \overline{\boldsymbol{u}}\|_{L^r(0,\mathcal{T};L^p(\Omega))}\big)\exp\big(C_3 \|\nabla \overline{\boldsymbol{u}}\|_{L^1(0,\mathcal{T};W^{1,p}(\Omega))}\big), \end{aligned}$$

where the constants C_2 and C_3 depend on Ω, p, r, $\mathfrak{We}$ and the norms of $\boldsymbol{G}_{\text{old}}$ and $\partial_s \boldsymbol{G}_{\text{old}}$. Their expressions will be detailed in the proof.

The existence of a unique solution to (4.2) follows from the application of the method of characteristics. Some details are given in [15] (Appendix p. 26) on Equation (2.10), that is, using the function $\boldsymbol{F}$. Note that the case presented here is a little bit more complicated. Equation (2.10) satisfied by $\boldsymbol{F}$ possesses a parameter $T \in \mathbb{R}$. This equation (2.10) is defined for time t such that $t \geq \max\{0, T\}$. Moreover the "initial" condition, with respect to the time t, depends on this parameter (see the initial conditions (2.13) and (2.14)). The characteristic method is applicable since conditions (2.13) and (2.14) exactly correspond to the boundary conditions for the "time" domain

$$\{(T, t) \in \mathbb{R}^2 \ ; \ t \geq \max\{0, T\}\}.$$

Note that in terms of the variables (s, t), the "time" domain is given by

$$\{(s, t) \in \mathbb{R}^2 \ ; \ s \geq 0 \ , \ t \geq 0\},$$

and that the boundary conditions in (4.2) are exactly given on the boundary of this domain. Finally these two boundary conditions are compatible since $\boldsymbol{G}_{\text{old}}|_{s=0} = \boldsymbol{\delta}$, see Remark 2.2.

In practice, the following estimates will be made on regular solution $\boldsymbol{G}_n$ which approaches the solution $\boldsymbol{G}$ when a regular velocity field $\boldsymbol{u}_n$ approaches the velocity $\boldsymbol{u}$. The regularity of these solutions $\boldsymbol{G}_n$ with respect to t and s comes from the Cauchy-Lipschitz theorem. For sake of simplicity, we omit the indexes "n". In the following proof, we refer to [15] for the passage to the limit $n \to +\infty$. The rest of the proof of Lemma 4.2 is split into three parts: in the first one (see Subsection 4.2.1) we obtain a first estimate concerning the regularity of $\boldsymbol{G}$, and in Subsection 4.2.3 we obtain the estimate for $\partial_t \boldsymbol{G}$. This estimate requires an estimate on $\partial_s \boldsymbol{G}$, which is given in Subsection 4.2.2. Note that we strongly use a Gronwall type lemma whose the proof is given in Appendix B.

4.2.1. Estimate for the deformation gradient $\boldsymbol{G}$. Let $p > d$ and take the scalar product of Equation (4.2) by $|\boldsymbol{G}|^{p-2}\boldsymbol{G}$. We deduce

$$\frac{1}{p}\partial_t\big(|\boldsymbol{G}|^p\big) + \frac{1}{\mathfrak{We}\,p}\partial_s\big(|\boldsymbol{G}|^p\big) + \frac{1}{p}\overline{\boldsymbol{u}}\cdot\nabla\big(|\boldsymbol{G}|^p\big) = |\boldsymbol{G}|^{p-2}(\boldsymbol{G}\cdot\nabla\overline{\boldsymbol{u}}) : \boldsymbol{G}.$$

Integrating for $\boldsymbol{x} \in \Omega$, due to the incompressible condition $\operatorname{div}\overline{\boldsymbol{u}} = 0$ and the homogeneous boundary Dirichlet condition for the velocity, we obtain

$$\begin{aligned}\partial_t\big(\|\boldsymbol{G}\|^p_{L^p(\Omega)}\big) + \frac{1}{\mathfrak{We}}\partial_s\big(\|\boldsymbol{G}\|^p_{L^p(\Omega)}\big) &= p\int_\Omega |\boldsymbol{G}|^{p-2}(\boldsymbol{G}\cdot\nabla\overline{\boldsymbol{u}}) : \boldsymbol{G}\\ &\le p\|\nabla\overline{\boldsymbol{u}}\|_{L^\infty(\Omega)}\|\boldsymbol{G}\|^p_{L^p(\Omega)}.\end{aligned}$$

We next use the continuous injection $W^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega)$, holds for $p > d$ and making appear a constant $C_0 = C_0(\Omega, p)$:

$$\partial_t\big(\|\boldsymbol{G}\|^p_{L^p(\Omega)}\big) + \frac{1}{\mathfrak{We}}\partial_s\big(\|\boldsymbol{G}\|^p_{L^p(\Omega)}\big) \le p\,C_0\|\nabla\overline{\boldsymbol{u}}\|_{W^{1,p}(\Omega)}\|\boldsymbol{G}\|^p_{L^p(\Omega)}. \tag{4.4}$$

Now, we take the spatial gradient in (4.2) and compute the scalar product of both sides of the resulting equation with $|\nabla\boldsymbol{G}|^{p-2}\nabla\boldsymbol{G}$ (we will note that this is a scalar product on the 3-tensor, defined by $A :: B = A_{i,j,k}B_{i,j,k}$). After integrating for $\boldsymbol{x} \in \Omega$ we obtain

$$\begin{aligned}\partial_t\big(\|\nabla\boldsymbol{G}\|^p_{L^p(\Omega)}\big) + \frac{1}{\mathfrak{We}}\partial_s\big(\|\nabla\boldsymbol{G}\|^p_{L^p(\Omega)}\big) &\le 2p\int_\Omega |\nabla\boldsymbol{G}|^p|\nabla\overline{\boldsymbol{u}}|\\ &\quad + p\int_\Omega |\boldsymbol{G}||\nabla\boldsymbol{G}|^{p-1}|\nabla^2\overline{\boldsymbol{u}}|.\end{aligned}$$

Using the Hölder inequality, we have

$$\begin{aligned}\partial_t\big(\|\nabla\boldsymbol{G}\|^p_{L^p(\Omega)}\big) + \frac{1}{\mathfrak{We}}\partial_s\big(\|\nabla\boldsymbol{G}\|^p_{L^p(\Omega)}\big) &\le 2p\|\nabla\overline{\boldsymbol{u}}\|_{L^\infty(\Omega)}\|\nabla\boldsymbol{G}\|^p_{L^p(\Omega)}\\ &\quad + p\|\boldsymbol{G}\|_{L^\infty(\Omega)}\|\nabla\boldsymbol{G}\|^{p-1}_{L^p(\Omega)}\|\nabla^2\overline{\boldsymbol{u}}\|_{L^p(\Omega)}.\end{aligned}$$

For $p > d$, using the continuous injection $W^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega)$ again, we deduce

$$\partial_t\big(\|\nabla\boldsymbol{G}\|^p_{L^p(\Omega)}\big) + \frac{1}{\mathfrak{We}}\partial_s\big(\|\nabla\boldsymbol{G}\|^p_{L^p(\Omega)}\big) \le 3p\,C_0\|\nabla\overline{\boldsymbol{u}}\|_{W^{1,p}(\Omega)}\|\boldsymbol{G}\|^p_{W^{1,p}(\Omega)}.$$

Adding this estimate with the estimate (4.4), we deduce

$$\partial_t\big(\|\boldsymbol{G}\|_{W^{1,p}(\Omega)}\big) + \frac{1}{\mathfrak{We}}\partial_s\big(\|\boldsymbol{G}\|_{W^{1,p}(\Omega)}\big) \le 3C_0\|\nabla\overline{\boldsymbol{u}}\|_{W^{1,p}(\Omega)}\|\boldsymbol{G}\|_{W^{1,p}(\Omega)}.$$

Using the initial conditions we have

$$\|\boldsymbol{G}\|_{W^{1,p}(\Omega)}\big|_{s=0} = \sqrt{d}\,|\Omega|^{\frac{1}{p}} \quad \text{and} \quad \|\boldsymbol{G}\|_{W^{1,p}(\Omega)}\big|_{t=0} = \|\boldsymbol{G}_{\text{old}}\|_{W^{1,p}(\Omega)},$$

the Gronwall type lemma (see Appendix B) implies that for $(s,t) \in \mathbb{R}^+ \times (0,\mathcal{T})$ we have

$$\|\boldsymbol{G}(s,t,\cdot)\|_{W^{1,p}(\Omega)} \leq \zeta(s,t)\exp\Big(3C_0 \int_0^t \|\nabla\overline{\boldsymbol{u}}\|_{W^{1,p}(\Omega)}\Big), \tag{4.5}$$

where $\zeta(s,t) = \begin{cases} \|\boldsymbol{G}_{\text{old}}\|_{W^{1,p}(\Omega)}\big(s - \dfrac{t}{\mathfrak{We}}\big) & \text{if } t \leq \mathfrak{We}\, s, \\ \sqrt{d}\,|\Omega|^{\frac{1}{p}} & \text{if } t > \mathfrak{We}\, s. \end{cases}$

The assumption $\boldsymbol{G}_{\text{old}} \in L^\infty(\mathbb{R}^+; W^{1,p}(\Omega))$ implies $\zeta \in L^\infty(\mathbb{R}^+ \times (0,\mathcal{T}))$ with

$$\|\zeta\|_{L^\infty(\mathbb{R}^+\times(0,\mathcal{T}))} \leq \max\big\{\|\boldsymbol{G}_{\text{old}}\|_{L^\infty(\mathbb{R}^+;W^{1,p}(\Omega))}, \sqrt{d}\,|\Omega|^{\frac{1}{p}}\big\}.$$

The relation (4.5) now reads

$$\begin{aligned} &\|\boldsymbol{G}\|_{L^\infty(\mathbb{R}^+\times(0,\mathcal{T});W^{1,p}(\Omega))} \\ &\qquad\qquad \leq \|\zeta\|_{L^\infty(\mathbb{R}^+\times(0,\mathcal{T}))} \exp\big(3C_0\|\nabla\overline{\boldsymbol{u}}\|_{L^1(0,\mathcal{T};W^{1,p}(\Omega))}\big). \end{aligned} \tag{4.6}$$

4.2.2. Estimate for the age derivate $\partial_s \boldsymbol{G}$. We first remark that the derivative $\boldsymbol{G}' = \partial_s \boldsymbol{G}$ exactly satisfies the same PDE as $\boldsymbol{G}$ (see the first equation of (4.2); that is due to the fact that $\overline{\boldsymbol{u}}$ does not depend on the variable s). We then deduce the same kind as (4.4):

$$\partial_t\big(\|\boldsymbol{G}'\|_{L^p(\Omega)}\big) + \frac{1}{\mathfrak{We}}\partial_s\big(\|\boldsymbol{G}'\|_{L^p(\Omega)}\big) \leq C_0\|\nabla\overline{\boldsymbol{u}}\|_{W^{1,p}(\Omega)}\|\boldsymbol{G}'\|_{L^p(\Omega)}.$$

But the initial conditions differ as follows:

$$\boldsymbol{G}'|_{t=0} = \partial_s \boldsymbol{G}_{\text{old}} \quad \text{and} \quad \boldsymbol{G}'|_{s=0} = \mathfrak{We}\,\nabla\overline{\boldsymbol{u}}.$$

This last condition is obtained using $s = 0$ in Equation (4.2). Note that this result is valid because we are working on regular solutions $\boldsymbol{G}_n$ (see the introduction of this proof) such that $\partial_t \boldsymbol{G}_n$ is continuous at $s = 0$. From Lemma B.1 given in Appendix B we obtain for all $(s,t) \in \mathbb{R}^+ \times (0,\mathcal{T})$ the estimate

$$\|\boldsymbol{G}'(s,t,\cdot)\|_{L^p(\Omega)} \leq \zeta'(s,t)\exp\Big(C_0 \int_0^t \|\nabla\overline{\boldsymbol{u}}\|_{W^{1,p}(\Omega)}\Big), \tag{4.7}$$

where $\zeta'(s,t) = \begin{cases} \|\partial_s\boldsymbol{G}_{\text{old}}\|_{L^p(\Omega)}\big(s - \dfrac{t}{\mathfrak{We}}\big) & \text{if } t \leq \mathfrak{We}\, s, \\ \mathfrak{We}\,\|\nabla\overline{\boldsymbol{u}}\|_{L^p(\Omega)}\big(t - \mathfrak{We}\, s\big) & \text{if } t > \mathfrak{We}\, s. \end{cases}$

For each $s \geq 0$, we estimate the $L^r(0,\mathcal{T})$-norm of the function $t \mapsto \zeta'(s,t)$ as follows: if $\mathcal{T} \leq \mathfrak{We}\, s$ then

$$\begin{aligned} \int_0^{\mathcal{T}} \zeta'(s,t)^r \,\mathrm{d}t &= \int_0^{\mathcal{T}} \|\partial_s\boldsymbol{G}_{\text{old}}\|^r_{L^p(\Omega)}\big(s - \frac{t}{\mathfrak{We}}\big)\,\mathrm{d}t \\ &= \mathfrak{We}\int_{s-\frac{\mathcal{T}}{\mathfrak{We}}}^{s} \|\partial_s\boldsymbol{G}_{\text{old}}\|^r_{L^p(\Omega)}(t')\,\mathrm{d}t' \\ &\leq \mathfrak{We}\,\|\partial_s\boldsymbol{G}_{\text{old}}\|^r_{L^r(\mathbb{R}^+;L^p(\Omega))}. \end{aligned}$$

If $\mathcal{T} > \mathfrak{We}\, s$ then

$$\begin{aligned}\int_0^{\mathcal{T}} \zeta'(s,t)^r \,\mathrm{d}t &= \int_0^{\mathfrak{We}\, s} \|\partial_s \boldsymbol{G}_{\mathrm{old}}\|_{L^p(\Omega)}^r \big(s - \frac{t}{\mathfrak{We}}\big)\,\mathrm{d}t \\ &\qquad + \mathfrak{We}^r \int_{\mathfrak{We}\, s}^{\mathcal{T}} \|\nabla \overline{\boldsymbol{u}}\|_{L^p(\Omega)}^r \big(t - \mathfrak{We}\, s\big)\,\mathrm{d}t \\ &= \mathfrak{We} \int_0^s \|\partial_s \boldsymbol{G}_{\mathrm{old}}\|_{L^p(\Omega)}^r (t')\,\mathrm{d}t' \\ &\qquad + \mathfrak{We}^r \int_0^{\mathcal{T} - \mathfrak{We}\, s} \|\nabla \overline{\boldsymbol{u}}\|_{L^p(\Omega)}^r (t')\,\mathrm{d}t' \\ &\le \mathfrak{We}\, \|\partial_s \boldsymbol{G}_{\mathrm{old}}\|_{L^r(\mathbb{R}^+;L^p(\Omega))}^r + \mathfrak{We}^r\, \|\nabla \overline{\boldsymbol{u}}\|_{L^r(0,\mathcal{T};L^p(\Omega))}^r.\end{aligned}$$

Finally, we obtain $\zeta' \in L^\infty(\mathbb{R}^+; L^r(0,\mathcal{T}))$ with

$$\|\zeta'\|_{L^\infty(\mathbb{R}^+;L^r(0,\mathcal{T}))} \le \mathfrak{We}^{\frac{1}{r}} \|\partial_s \boldsymbol{G}_{\mathrm{old}}\|_{L^r(\mathbb{R}^+;L^p(\Omega))} + \mathfrak{We} \|\nabla \overline{\boldsymbol{u}}\|_{L^r(0,\mathcal{T};L^p(\Omega))}.$$

The relation (4.7) now reads

$$\begin{aligned}&\|\boldsymbol{G}'\|_{L^\infty(\mathbb{R}^+;L^r(0,\mathcal{T};L^p(\Omega)))} \\ &\qquad\qquad \le \|\zeta'\|_{L^\infty(\mathbb{R}^+;L^r(0,\mathcal{T}))} \exp\big(C_0 \|\nabla \overline{\boldsymbol{u}}\|_{L^1(0,\mathcal{T};W^{1,p}(\Omega))}\big).\end{aligned} \tag{4.8}$$

4.2.3. Estimate for the time derivate $\partial_t \boldsymbol{G}$. Isolating the term $\partial_t \boldsymbol{G}$ in Equation (4.2) we have

$$\begin{aligned}\|\partial_t \boldsymbol{G}\|_{L^p(\Omega)} &\le \frac{1}{\mathfrak{We}} \|\boldsymbol{G}'\|_{L^p(\Omega)} + \|\overline{\boldsymbol{u}}\|_{L^\infty(\Omega)} \|\nabla \boldsymbol{G}\|_{L^p(\Omega)} + \|\boldsymbol{G}\|_{L^\infty(\Omega)} \|\nabla \overline{\boldsymbol{u}}\|_{L^p(\Omega)} \\ &\le \frac{1}{\mathfrak{We}} \|\boldsymbol{G}'\|_{L^p(\Omega)} + C_0 \|\overline{\boldsymbol{u}}\|_{W^{1,p}(\Omega)} \|\nabla \boldsymbol{G}\|_{L^p(\Omega)} \\ &\qquad\qquad + C_0 \|\boldsymbol{G}\|_{W^{1,p}(\Omega)} \|\nabla \overline{\boldsymbol{u}}\|_{L^p(\Omega)}.\end{aligned}$$

Introducing the Poincaré inequality with a constant $C_P = C_P(\Omega, p)$, which holds since $\overline{\boldsymbol{u}}$ vanishes on the boundary of the domain, we obtain

$$\|\partial_t \boldsymbol{G}\|_{L^p(\Omega)} \le \frac{1}{\mathfrak{We}} \|\boldsymbol{G}'\|_{L^p(\Omega)} + C_0 (1 + C_P) \|\nabla \overline{\boldsymbol{u}}\|_{L^p(\Omega)} \|\boldsymbol{G}\|_{W^{1,p}(\Omega)}.$$

Taking the $L^r(0,\mathcal{T})$-norm in the variable t, and next the $L^\infty(\mathbb{R}^+)$-norm in the variable s, we obtain

$$\begin{aligned}\|\partial_t \boldsymbol{G}\|_{L^\infty(\mathbb{R}^+;L^r(0,\mathcal{T};L^p(\Omega)))} &\le \frac{1}{\mathfrak{We}} \|\boldsymbol{G}'\|_{L^\infty(\mathbb{R}^+;L^r(0,\mathcal{T};L^p(\Omega)))} \\ &\qquad + C_0 (1 + C_P) \|\nabla \overline{\boldsymbol{u}}\|_{L^r(0,\mathcal{T};L^p(\Omega))} \|\boldsymbol{G}\|_{L^\infty(\mathbb{R}^+ \times (0,\mathcal{T});W^{1,p}(\Omega))}.\end{aligned}$$

Using the previous estimates (4.6) and (4.8), we deduce the result announced in Lemma 4.2.

4.3. Estimates for the stress tensor $\boldsymbol{\tau}$

Lemma 4.3. *Let $\mathcal{T} > 0$, $r \in]1, +\infty[$ and $p \in]d, +\infty[$.*
If $\overline{\boldsymbol{G}} \in L^\infty(\mathbb{R}^+ \times (0,\mathcal{T}); W^{1,p}(\Omega))$ and $\partial_t \overline{\boldsymbol{G}} \in L^\infty(\mathbb{R}^+; L^r(0,\mathcal{T}; L^p(\Omega)))$ then the stress tensor $\boldsymbol{\tau}$ defined by the integral relation (4.3) *belongs to the space $L^\infty(0,\mathcal{T}; W^{1,p}(\Omega))$ and its time derivative $\partial_t \boldsymbol{\tau}$ belongs to $L^s(0,\mathcal{T}; L^p(\Omega))$.*

Moreover, there exists a continuous increasing function $F_0 : \mathbb{R}^+ \mapsto \mathbb{R}^+$ *such that*

$$\begin{aligned}\|\boldsymbol{\tau}\|_{L^\infty(0,\mathcal{T};W^{1,p}(\Omega))} &+ \|\partial_t\boldsymbol{\tau}\|_{L^r(0,\mathcal{T};L^p(\Omega))}\\ &\leq \frac{\omega}{\mathfrak{We}} F_0\big(\|\overline{\boldsymbol{G}}\|_{L^\infty(\mathbb{R}^+\times(0,\mathcal{T});W^{1,p}(\Omega))} + \|\partial_t\overline{\boldsymbol{G}}\|_{L^\infty(\mathbb{R}^+;L^r(0,\mathcal{T};L^p(\Omega)))}\big).\end{aligned}$$

The function F_0 *depends on* Ω, p *and on the growth of the function* $\mathcal{S}$.

Proof. Since the function $\mathcal{S}$ is of class $\mathcal{C}^1$, we can introduce the following continuous and non-decreasing real functions

$$\mathcal{S}_0 : c \in \mathbb{R}^+ \longmapsto \max_{|\boldsymbol{G}|\leq c} |\mathcal{S}(\boldsymbol{G})| \in \mathbb{R}^+,$$
$$\mathcal{S}_1 : c \in \mathbb{R}^+ \longmapsto \max_{|\boldsymbol{G}|\leq c} |\mathcal{S}'(\boldsymbol{G})| \in \mathbb{R}^+,$$

where the derivative $\mathcal{S}'(\boldsymbol{G})$ denotes the 4-tensor whose the coefficient (i,j,k,ℓ), denoted $\partial_{(ij)}\mathcal{S}(\overline{\boldsymbol{G}})_{k\ell}$, is the derivative of $\big(\mathcal{S}(\boldsymbol{G})\big)_{k\ell}$ with respect to the tensor $\boldsymbol{E}_{ij}$ of the canonical basis of the space $\mathcal{L}(\mathbb{R}^d)$ of real $d\times d$ matrices, see Appendix A.
Due to the continuous injection $W^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega)$, the function $\overline{\boldsymbol{G}}$ introduced in the hypothesis of the lemma is bounded in $\mathbb{R}^+\times[0,\mathcal{T}]\times\Omega$ and we have

$$\begin{aligned}\|\mathcal{S}(\overline{\boldsymbol{G}})\|_{L^\infty(\mathbb{R}^+\times(0,\mathcal{T})\times\Omega)} &\leq \mathcal{S}_0\big(\|\overline{\boldsymbol{G}}\|_{L^\infty(\mathbb{R}^+\times(0,\mathcal{T})\times\Omega)}\big)\\ &\leq \mathcal{S}_0\big(C_0\,\|\overline{\boldsymbol{G}}\|_{L^\infty(\mathbb{R}^+\times(0,\mathcal{T});W^{1,p}(\Omega))}\big).\end{aligned}$$

To simplify, we note $\overline{c} := C_0\,\|\overline{\boldsymbol{G}}\|_{L^\infty(\mathbb{R}^+\times(0,\mathcal{T});W^{1,p}(\Omega))}$. In the same way the function $\mathcal{S}'(\overline{\boldsymbol{G}})$ is bounded by the real $\mathcal{S}_1(\overline{c})$.

$\boldsymbol{L^p}$-norm for $\boldsymbol{\tau}$ – We easily have the following bound for the stress tensor $\boldsymbol{\tau}$ given by the formula (4.3): $|\boldsymbol{\tau}(t,\boldsymbol{x})| \leq \frac{\omega}{\mathfrak{We}}\mathcal{S}_0(\overline{c})$ for a. e. $(t,\boldsymbol{x}) \in (0,\mathcal{T})\times\Omega$. We note that we used $\int_0^\infty m = 1$. We deduce in particular that

$$\|\boldsymbol{\tau}\|_{L^\infty(0,\mathcal{T};L^p(\Omega))} \leq |\Omega|^{\frac{1}{p}}\,\frac{\omega}{\mathfrak{We}}\mathcal{S}_0(\overline{c}). \tag{4.9}$$

$\boldsymbol{W^{1,p}}$-norm for $\boldsymbol{\tau}$ – Taking the spatial gradient of the expression (4.3) given the stress tensor we obtain

$$\nabla\boldsymbol{\tau}(t,\boldsymbol{x}) = \frac{\omega}{\mathfrak{We}}\int_0^{+\infty} m(s)\nabla\overline{\boldsymbol{G}}(s,t,\boldsymbol{x}) : \mathcal{S}'(\overline{\boldsymbol{G}}(s,t,\boldsymbol{x}))\,\mathrm{d}s.$$

The meaning of the symbols here is the following. Component by component, the equality above written

$$\begin{aligned}\big[\nabla\boldsymbol{\tau}(t,\boldsymbol{x})\big]_{ijk} &= \partial_i\boldsymbol{\tau}_{jk}(t,\boldsymbol{x})\\ &= \frac{\omega}{\mathfrak{We}}\int_0^{+\infty} m(s)\sum_{\ell,m}\partial_i\overline{\boldsymbol{G}}_{\ell m}(s,t,\boldsymbol{x})\,\partial_{(\ell m)}\mathcal{S}\big(\overline{\boldsymbol{G}}(s,t,\boldsymbol{x})\big)_{jk}\,\mathrm{d}s.\end{aligned}$$

Using the Hölder inequality, the L^∞-bound on $\mathcal{S}'(\overline{\boldsymbol{G}})$ and $\int_0^\infty m = 1$, we obtain

$$\begin{aligned}|\nabla\boldsymbol{\tau}(t,\boldsymbol{x})|^p &= \frac{\omega^p}{\mathfrak{We}^p}\Big|\int_0^\infty m(s)^{\frac{1}{p}}\nabla\overline{\boldsymbol{G}}(s,t,\boldsymbol{x}) : m(s)^{1-\frac{1}{p}}\mathcal{S}'\big(\overline{\boldsymbol{G}}(s,t,\boldsymbol{x})\big)\,\mathrm{d}s\Big|^p\\ &\leq \frac{\omega^p}{\mathfrak{We}^p}\mathcal{S}_1(\overline{c})^p\int_0^\infty m(s)|\nabla\overline{\boldsymbol{G}}(s,t,\boldsymbol{x})|^p\,\mathrm{d}s.\end{aligned}$$

Integrating for $\boldsymbol{x}\in\Omega$, we obtain

$$\|\nabla\boldsymbol{\tau}(t,\cdot)\|^p_{L^p(\Omega)} \leq \frac{\omega^p}{\mathfrak{We}^p}\mathcal{S}_1(\overline{c})^p\int_0^\infty m(s)\|\nabla\overline{\boldsymbol{G}}(s,t,\cdot)\|^p_{L^p(\Omega)}\,\mathrm{d}s.$$

Due to the definition of the bound $\overline{c}$, this implies

$$\|\nabla\boldsymbol{\tau}\|_{L^\infty(0,\mathcal{T};L^p(\Omega))} \leq \frac{\omega}{\mathfrak{We}}\mathcal{S}_1(\overline{c})\,\frac{\overline{c}}{C_0}. \tag{4.10}$$

$\boldsymbol{L^p}$-norm for $\boldsymbol{\partial_t\tau}$ – Similarly, we obtain a bound for $\partial_t\boldsymbol{\tau}$ in $L^p(\Omega)$: we have

$$\partial_t\boldsymbol{\tau}(t,\boldsymbol{x}) = \frac{\omega}{\mathfrak{We}}\int_0^{+\infty} m(s)\partial_t\overline{\boldsymbol{G}}(s,t,\boldsymbol{x}) : \mathcal{S}'\big(\overline{\boldsymbol{G}}(s,t,\boldsymbol{x})\big)\,\mathrm{d}s.$$

Using the Hölder inequality to control the quantity $|\partial_t\boldsymbol{\tau}(t,\boldsymbol{x})|^p$, and next an integration for $\boldsymbol{x}\in\Omega$, we obtain

$$\|\partial_t\boldsymbol{\tau}(t,\cdot)\|^p_{L^p(\Omega)} \leq \frac{\omega^p}{\mathfrak{We}^p}\mathcal{S}_1(\overline{c})^p\int_0^\infty m(s)\|\partial_t\overline{\boldsymbol{G}}(s,t,\cdot)\|^p_{L^p(\Omega)}\,\mathrm{d}s.$$

Due to the assumption on $\partial_t\boldsymbol{G}$, that is $\widetilde{c} := \|\partial_t\overline{\boldsymbol{G}}\|_{L^\infty(\mathbb{R}^+;L^r(0,\mathcal{T};L^p(\Omega)))} < +\infty$, we deduce

$$\|\partial_t\boldsymbol{\tau}\|_{L^r(0,\mathcal{T};L^p(\Omega))} \leq \frac{\omega}{\mathfrak{We}}\mathcal{S}_1(\overline{c})\,\widetilde{c}. \tag{4.11}$$

The estimates (4.9), (4.10) and (4.11) show that $\boldsymbol{\tau}$ and $\partial_t\boldsymbol{\tau}$ are bounded respectively in $L^\infty(0,\mathcal{T};W^{1,p}(\Omega))$ and in $L^r(0,\mathcal{T};L^p(\Omega))$, and that these bounds continuously depend on $\overline{c}$ and $\widetilde{c}$, and increase in both variables. □

4.4. Proof of Theorem 3.1

For any $\mathcal{T}>0$ we introduce the Banach space

$$\mathscr{B}(\mathcal{T}) = L^r(0,\mathcal{T};W^{1,p}_0(\Omega))\times\mathcal{C}(\mathbb{R}^+\times[0,\mathcal{T}];L^p(\Omega))\times\mathcal{C}([0,\mathcal{T}];L^p(\Omega))$$

and for any $R_1>0$, $R_2>0$ and $R_3>0$ the subset

$$\begin{aligned}\mathscr{H}(\mathcal{T},R_1,R_2,R_3) = \Big\{&(\overline{\boldsymbol{u}},\overline{\boldsymbol{G}},\overline{\boldsymbol{\tau}})\in\mathscr{B}(\mathcal{T})\ ;\\ &\overline{\boldsymbol{u}}\in L^r(0,\mathcal{T};D(A_p)),\quad \partial_t\overline{\boldsymbol{u}}\in L^r(0,\mathcal{T};H_p),\\ &\overline{\boldsymbol{G}}\in L^\infty(\mathbb{R}^+\times(0,\mathcal{T});W^{1,p}(\Omega)),\quad \partial_s\overline{\boldsymbol{G}},\ \partial_t\overline{\boldsymbol{G}}\in L^\infty(\mathbb{R}^+;L^r(0,\mathcal{T};L^p(\Omega))),\\ &\overline{\boldsymbol{\tau}}\in L^\infty(0,\mathcal{T};W^{1,p}(\Omega)),\quad \partial_t\overline{\boldsymbol{\tau}}\in L^r(0,\mathcal{T};L^p(\Omega)),\\ &\overline{\boldsymbol{u}}|_{t=0}=\boldsymbol{u}_0,\quad \overline{\boldsymbol{G}}|_{t=0}=\boldsymbol{G}_{\mathrm{old}},\quad \overline{\boldsymbol{G}}|_{s=0}=\boldsymbol{\delta},\\ &\|\overline{\boldsymbol{u}}\|_{L^r(0,\mathcal{T};W^{2,p}(\Omega))}+\|\partial_t\overline{\boldsymbol{u}}\|_{L^r(0,\mathcal{T};L^p(\Omega))}\leq R_1,\\ &\|\overline{\boldsymbol{G}}\|_{L^\infty(\mathbb{R}^+\times(0,\mathcal{T});W^{1,p}(\Omega))}+\|\partial_s\overline{\boldsymbol{G}},\,\partial_t\overline{\boldsymbol{G}}\|_{L^\infty(\mathbb{R}^+;L^r(0,\mathcal{T};L^p(\Omega)))}\leq R_2,\\ &\|\overline{\boldsymbol{\tau}}\|_{L^\infty(0,\mathcal{T};W^{1,p}(\Omega))}+\|\partial_t\overline{\boldsymbol{\tau}}\|_{L^r(0,\mathcal{T};L^p(\Omega))}\leq R_3\ \Big\}.\end{aligned}$$

Remark 4.4. Such a set is non-empty, for instance if R_1 and R_2 are large enough. More precisely, if

$$R_1 \geq \frac{C_1}{1-\omega}\|\boldsymbol{u}_0\|_{D_p^r(\Omega)} \qquad \text{and} \qquad R_2 \geq \|\boldsymbol{G}_{\mathrm{old}}\|_{L^\infty(\mathbb{R}^+;W^{1,p}(\Omega))} \tag{4.12}$$

then for any $\mathcal{T} > 0$ and any $R_3 > 0$ we can build a velocity field $\boldsymbol{u}^\star$ such that $(\boldsymbol{u}^\star, \boldsymbol{G}_{\mathrm{old}}, \mathbf{0}) \in \mathscr{H}(\mathcal{T}, R_1, R_2, R_3)$, see an example of construction in [15, 21].

Remark 4.5. If $(\boldsymbol{u}, \boldsymbol{G}, \boldsymbol{\tau}) \in \mathscr{H}(\mathcal{T}, R_1, R_2, R_3)$ for some $\mathcal{T}$, R_1, R_2 and R_3 then the velocity field $\boldsymbol{u}$ and the tensor $\boldsymbol{G}$ are continuous with respect to the time t and the age s. In fact, these continuity properties follow from the Sobolev injections of kind $W^{1,\alpha}(0,A;X) \subset \mathcal{C}([0,A];X)$, holds for $\alpha > 1$. Moreover, they make sense of the initial conditions $\boldsymbol{u}|_{t=0} = \boldsymbol{u}_0$, $\boldsymbol{G}|_{t=0} = \boldsymbol{G}_{\mathrm{old}}$ and $\boldsymbol{G}|_{s=0} = \boldsymbol{\delta}$.

We consider the mapping

$$\begin{aligned} \Phi \ : \ \mathscr{H}(\mathcal{T}, R_1, R_2, R_3) &\longrightarrow \ \mathscr{B}(\mathcal{T}) \\ (\overline{\boldsymbol{u}}, \overline{\boldsymbol{G}}, \overline{\boldsymbol{\tau}}) &\longmapsto \ (\boldsymbol{u}, \boldsymbol{G}, \boldsymbol{\tau}), \end{aligned}$$

where $\boldsymbol{u}$ is the unique solution of the Stokes problem (4.1) with

$$\overline{\boldsymbol{g}} = -\mathfrak{Re}\,\overline{\boldsymbol{u}} \cdot \nabla \overline{\boldsymbol{u}} + \operatorname{div} \overline{\boldsymbol{\tau}} + \boldsymbol{f}; \tag{4.13}$$

where $\boldsymbol{G}$ solves the problem (4.2) and where $\boldsymbol{\tau}$ is given by the integral formula (4.3). The goal of this proof is to show that the application Φ has a fixed point. For this we first prove that Φ leaves a set $\mathscr{H}(\mathcal{T}, R_1, R_2, R_3)$ invariant (for a “good” choice of $\mathcal{T}$, R_1, R_2 and R_3).
Let $\mathcal{T} > 0$, $R_1 > 0$, $R_2 > 0$, $R_3 > 0$ and $(\overline{\boldsymbol{u}}, \overline{\boldsymbol{G}}, \overline{\boldsymbol{\tau}}) \in \mathscr{H}(\mathcal{T}, R_1, R_2, R_3)$. If we denote by $(\boldsymbol{u}, \boldsymbol{G}, \boldsymbol{\tau}) = \Phi(\overline{\boldsymbol{u}}, \overline{\boldsymbol{G}}, \overline{\boldsymbol{\tau}})$, we will show that the previous lemmas imply estimates of $(\boldsymbol{u}, \boldsymbol{G}, \boldsymbol{\tau})$ with respect to the norms of $(\overline{\boldsymbol{u}}, \overline{\boldsymbol{G}}, \overline{\boldsymbol{\tau}})$, that is with respect to $(\mathcal{T}, R_1, R_2, R_3)$.

Velocity estimate – From Lemma 4.1 we can estimate $\boldsymbol{u}$ and $\partial_t \boldsymbol{u}$ using the norm $\|\boldsymbol{g}\|_{L^r(0,\mathcal{T};L^p(\Omega))}$. For the source term $\boldsymbol{g}$ given by the relation (4.13), we have

$$\begin{aligned} \|\boldsymbol{g}\|_{L^r(0,\mathcal{T};L^p(\Omega))} \leq \mathfrak{Re} & \underbrace{\|\overline{\boldsymbol{u}} \cdot \nabla \overline{\boldsymbol{u}}\|_{L^r(0,\mathcal{T};L^p(\Omega))}}_{T_1} \\ & + \underbrace{\|\overline{\boldsymbol{\tau}}\|_{L^r(0,\mathcal{T};W^{1,p}(\Omega))}}_{T_2} + \|\boldsymbol{f}\|_{L^r(0,\mathcal{T};L^p(\Omega))}. \end{aligned}$$

Since we have the bound R_3 on $\overline{\boldsymbol{\tau}}$ in $L^\infty(0,\mathcal{T};W^{1,p}(\Omega))$, the term T_2 satisfies $T_2 \leq \mathcal{T}^{\frac{1}{r}} R_3$. The bilinear term T_1 is more difficult to estimate. We follow the ideas of [15] and we generalize their result to the d-dimensional case (the paper [15] only deals with the case $d = 3$):

$$\begin{aligned} T_1 &\leq \|\overline{\boldsymbol{u}}\|_{L^{2r}(0,\mathcal{T};L^\infty(\Omega))} \|\nabla \overline{\boldsymbol{u}}\|_{L^{2r}(0,\mathcal{T};L^p(\Omega))} \\ &\leq \mathcal{T}^{\frac{p-d}{2rp}} \|\overline{\boldsymbol{u}}\|_{L^{\frac{2rp}{d}}(0,\mathcal{T};L^\infty(\Omega))} \|\overline{\boldsymbol{u}}\|_{L^{2r}(0,\mathcal{T};W^{1,p}(\Omega))}. \end{aligned} \tag{4.14}$$

But we have the following estimate (see [17]):

$$\|\overline{\boldsymbol{u}}\|_{L^\infty(\Omega)} \le C\|\overline{\boldsymbol{u}}\|_{L^p(\Omega)}^{\frac{p-d}{p}}\|\overline{\boldsymbol{u}}\|_{W^{1,p}(\Omega)}^{\frac{d}{p}},$$

which, after integrating with respect to time, implies

$$\|\overline{\boldsymbol{u}}\|_{L^{\frac{2pr}{d}}(0,\mathcal{T};L^\infty(\Omega))} \le C\|\overline{\boldsymbol{u}}\|_{L^\infty(0,\mathcal{T};L^p(\Omega))}^{\frac{p-d}{p}}\|\overline{\boldsymbol{u}}\|_{L^{2r}(0,\mathcal{T};W^{1,p}(\Omega))}^{\frac{d}{p}}. \tag{4.15}$$

Note that the constant C introduced here only depends on Ω, p and d. Moreover, by interpolation, we have

$$\|\overline{\boldsymbol{u}}\|_{L^{2r}(0,\mathcal{T};W^{1,p}(\Omega))} \le \|\overline{\boldsymbol{u}}\|_{L^\infty(0,\mathcal{T};L^p(\Omega))}^{\frac{1}{2}}\|\overline{\boldsymbol{u}}\|_{L^r(0,\mathcal{T};W^{2,p}(\Omega))}^{\frac{1}{2}}. \tag{4.16}$$

Using (4.15) and (4.16), the estimate (4.14) now reads

$$T_1 \le C\,\mathcal{T}^{\frac{p-d}{2rp}}\|\overline{\boldsymbol{u}}\|_{L^\infty(0,\mathcal{T};L^p(\Omega))}^{\frac{3p-d}{2p}}\|\overline{\boldsymbol{u}}\|_{L^r(0,\mathcal{T};W^{2,p}(\Omega))}^{\frac{p+d}{2p}}. \tag{4.17}$$

Finally, we use $\overline{\boldsymbol{u}}(t,\boldsymbol{x}) = \boldsymbol{u}_0(\boldsymbol{x}) + \int_0^t \partial_t\overline{\boldsymbol{u}}$ to obtain

$$\begin{aligned}\|\overline{\boldsymbol{u}}\|_{L^\infty(0,\mathcal{T};L^p(\Omega))} &\le \|\boldsymbol{u}_0\|_{L^p(\Omega)} + \|\partial_t\overline{\boldsymbol{u}}\|_{L^1(0,\mathcal{T};L^p(\Omega))}\\ &\le \|\boldsymbol{u}_0\|_{L^p(\Omega)} + \mathcal{T}^{1-\frac{1}{r}}\|\partial_t\overline{\boldsymbol{u}}\|_{L^r(0,\mathcal{T};L^p(\Omega))}.\end{aligned}$$

Using the bound R_1 for $\overline{\boldsymbol{u}}$ and $\partial_t\overline{\boldsymbol{u}}$ given in the definition of $\mathscr{H}(\mathcal{T},R_1,R_2,R_3)$, the estimate (4.17) becomes

$$T_1 \le C\,\mathcal{T}^{\frac{p-d}{2rp}}R_1^{\frac{p+d}{2p}}\|\boldsymbol{u}_0\|_{L^p(\Omega)}^{\frac{3p-d}{2p}} + C\,\mathcal{T}^{\frac{3p-d}{2p}-\frac{1}{r}}R_1^2.$$

We now use this bound to control the source term $\boldsymbol{g}$. Lemma 4.1 implies:

$$\begin{aligned}\|\boldsymbol{u}\|_{L^r(0,\mathcal{T};W^{2,p}(\Omega))} + \|\partial_t\boldsymbol{u}\|_{L^r(0,\mathcal{T};L^p(\Omega))} \le \frac{C_1}{1-\omega}\Big(\|\boldsymbol{u}_0\|_{D_p^r(\Omega)}\\ + \|\boldsymbol{f}\|_{L^r(0,\mathcal{T};L^p(\Omega))} + \mathfrak{Re}\,C\,\mathcal{T}^\alpha R_1^\beta\|\boldsymbol{u}_0\|_{D_p^r(\Omega)}^\gamma + \mathfrak{Re}\,C\,\mathcal{T}^\delta R_1^2 + \mathcal{T}^{\frac{1}{r}}R_3\Big),\end{aligned} \tag{4.18}$$

where the assumptions $p>d$ and $r>1$ imply that α, β, γ and δ are positive numbers. This estimate (4.18) can be rewrite as

$$\|\boldsymbol{u}\|_{L^r(0,\mathcal{T};W^{2,p}(\Omega))}+\|\partial_t\boldsymbol{u}\|_{L^r(0,\mathcal{T};L^p(\Omega))} \le \widetilde{C_1}\Big(1+\mathcal{T}^{\frac{1}{r}}\,R_3+K(\mathcal{T},R_1)\Big), \tag{4.19}$$

where $\widetilde{C_1}$ may also depends on ω and on the norm of $\boldsymbol{u}_0$ and $\boldsymbol{f}$ in their spaces. It is important to notice that for each $R_1>0$ we have $\lim\limits_{\mathcal{T}\to 0} K(\mathcal{T},R_1) = 0$, and that $\widetilde{C_1} \ge \frac{C_1}{1-\omega}\|\boldsymbol{u}_0\|_{D_p^r(\Omega)}$.

Deformation gradient estimate – From Lemma 4.2, we have

$$\begin{aligned}\|\boldsymbol{G}\|_{L^\infty(\mathbb{R}^+\times(0,\mathcal{T});W^{1,p}(\Omega))} + \|\partial_t\boldsymbol{G}\|_{L^\infty(\mathbb{R}^+;L^r(0,\mathcal{T};L^p(\Omega)))}\\ \le C_2\big(1+R_1\big)\exp\big(C_3\mathcal{T}^{1-\frac{1}{r}}R_1\big).\end{aligned} \tag{4.20}$$

Stress tensor estimate – From Lemma 4.3, we have

$$\|\boldsymbol{\tau}\|_{L^\infty(0,\mathcal{T};W^{1,p}(\Omega))} + \|\partial_t\boldsymbol{\tau}\|_{L^r(0,\mathcal{T};L^p(\Omega))} \le \frac{\omega}{\mathfrak{We}}F_0(R_2). \tag{4.21}$$

Φ-Invariant subset – If we then successively choose

$$
\begin{aligned}
&R_1^\star = 2\widetilde{C_1},\\
&R_2^\star = C_2\big(1+R_1^\star\big)\exp\big(C_3 R_1^\star\big) + \|\boldsymbol{G}_{\mathrm{old}}\|_{L^\infty(\mathbb{R}^+;W^{1,p}(\Omega))},\\
&R_3^\star = \frac{\omega}{\mathfrak{We}} F_0(R_2^\star),\\
&\text{and} \quad \mathcal{T}_\star \leq 1 \text{ small enough to have } \mathcal{T}_\star^{\frac{1}{r}} R_3^\star + K(\mathcal{T}_\star, R_1^\star) \leq 1,
\end{aligned}
$$

then we verify that $\mathscr{H}(\mathcal{T}_\star, R_1^\star, R_2^\star, R_3^\star) \neq \emptyset$ (that is the inequalities (4.12) hold). For such a choice, the estimates (4.19), (4.20) and (4.21) imply that $\Phi(\mathscr{H}(\mathcal{T}_\star, R_1^\star, R_2^\star, R_3^\star)) \subset \mathscr{H}(\mathcal{T}_\star, R_1^\star, R_2^\star, R_3^\star)$. Moreover the function Φ is continuous and $\mathscr{H}(\mathcal{T}_\star, R_1^\star, R_2^\star, R_3^\star)$ is a convex compact subset of $\mathscr{B}(\mathcal{T}_\star)$, see [21] for similar properties. We conclude the proof using the Schauder's theorem. □

5. Proof for the uniqueness result

This section is devoted to the proof of the local existence theorem 3.2.
As usual, we take the difference of the two solutions indexed by 1 and 2. The vector $\boldsymbol{u} = \boldsymbol{u}_1 - \boldsymbol{u}_2$, the scalar $p = p_1 - p_2$ and the tensors $\boldsymbol{\tau} = \boldsymbol{\tau}_1 - \boldsymbol{\tau}_2$, $\boldsymbol{G} = \boldsymbol{G}_1 - \boldsymbol{G}_2$ satisfy the following:

$$
\begin{cases}
\mathfrak{Re}(\partial_t \boldsymbol{u} + \boldsymbol{u}_1\cdot\nabla\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}_2) + \nabla p - (1-\omega)\Delta\boldsymbol{u} = \operatorname{div}\boldsymbol{\tau},\\
\operatorname{div}\boldsymbol{u} = 0,\\
\boldsymbol{\tau} = \dfrac{\omega}{\mathfrak{We}}\displaystyle\int_0^{+\infty} m(s)\Big[\mathcal{S}\big(\boldsymbol{G}_1(s,\cdot,\cdot)\big) - \mathcal{S}\big(\boldsymbol{G}_2(s,\cdot,\cdot)\big)\Big]\,\mathrm{d}s,\\
\partial_t\boldsymbol{G} + \dfrac{1}{\mathfrak{We}}\partial_s\boldsymbol{G} + \boldsymbol{u}_1\cdot\nabla\boldsymbol{G} + \boldsymbol{u}\cdot\nabla\boldsymbol{G}_2 = \boldsymbol{G}_1\cdot\nabla\boldsymbol{u} + \boldsymbol{G}\cdot\nabla\boldsymbol{u}_2,
\end{cases}
\tag{5.1}
$$

together with zero initial conditions $\boldsymbol{u}\big|_{t=0} = \boldsymbol{0}$ and $\boldsymbol{G}\big|_{t=0} = \boldsymbol{G}\big|_{s=0} = \boldsymbol{0}$. Note that the regularity of $\boldsymbol{G}_i$ and the definition of the stress tensor $\boldsymbol{\tau}_i$ implies that $\boldsymbol{\tau}_i \in L^\infty(0,\mathcal{T};W^{1,d}(\Omega))$ (the proof is similar that those presented in the proof of the existence theorem 3.1). The uniqueness proof consists in demonstrate that $\boldsymbol{u} = \boldsymbol{0}$ and that $\boldsymbol{G} = \boldsymbol{\tau} = \boldsymbol{0}$. We will initially provide estimates on these three quantities.

Velocity estimate – Taking the scalar product of the first equation of System (5.1) by $\boldsymbol{u}$ in $L^2(\Omega)$, we obtain

$$
\frac{\mathfrak{Re}}{2} d_t\big(\|\boldsymbol{u}\|^2_{L^2(\Omega)}\big) + (1-\omega)\|\nabla\boldsymbol{u}\|^2_{L^2(\Omega)} = -\int_\Omega \boldsymbol{\tau}\cdot\nabla\boldsymbol{u} - \mathfrak{Re}\int_\Omega (\boldsymbol{u}\cdot\nabla\boldsymbol{u}_2)\cdot\boldsymbol{u}.
$$

From the Hölder inequality and the Young inequality, we obtain

$$
\begin{aligned}
\mathfrak{Re}\, d_t\big(\|\boldsymbol{u}\|^2_{L^2(\Omega)}\big) + (1-\omega)\|\nabla\boldsymbol{u}\|^2_{L^2(\Omega)} \leq{}& \frac{4}{1-\omega}\|\boldsymbol{\tau}\|^2_{L^2(\Omega)}\\
&+ \mathfrak{Re}\|\nabla\boldsymbol{u}_2\|_{L^\infty(\Omega)}\|\boldsymbol{u}\|^2_{L^2(\Omega)}.
\end{aligned}
$$

Introducing $Z(t) = \mathfrak{Re}\,\|\boldsymbol{u}\|^2_{L^2(\Omega)}$ and $C_Z(t) = \|\nabla \boldsymbol{u}_2\|_{L^\infty(\Omega)} \in L^1(0,\mathcal{T})$, this estimate reads

$$Z'(t) + (1-\omega)\|\nabla \boldsymbol{u}\|^2_{L^2(\Omega)} \le \frac{4}{1-\omega}\|\boldsymbol{\tau}\|^2_{L^2(\Omega)} + C_Z(t)Z(t). \tag{5.2}$$

Stress tensor estimate – From the definition of the stress tensor $\boldsymbol{\tau}$ in the System (5.1) we can use the Cauchy-Schwarz inequality to deduce that for all $(t,\boldsymbol{x}) \in (0,\mathcal{T})\times\Omega$:

$$|\boldsymbol{\tau}(t,\boldsymbol{x})|^2 \le \frac{\omega^2}{\mathfrak{We}^2}\int_0^{+\infty} m(s)\left|\mathcal{S}\big(\boldsymbol{G}_1(s,t,\boldsymbol{x})\big) - \mathcal{S}\big(\boldsymbol{G}_2(s,t,\boldsymbol{x})\big)\right|^2 \mathrm{d}s.$$

By assumption, the function $\mathcal{S}$ is of class $\mathcal{C}^1$, so that $\mathcal{S}'$ is bounded on each compact. Since $\boldsymbol{G}_i$, $i\in\{1,2\}$, belongs to $L^\infty(\mathbb{R}^+\times(0,\mathcal{T})\times\Omega)$ we deduce that there exists a constant C', only depending on the norm $\|\boldsymbol{G}_i\|_{L^\infty(\mathbb{R}^+\times(0,\mathcal{T})\times\Omega)}$ such that $|\mathcal{S}(\boldsymbol{G}_1)-\mathcal{S}(\boldsymbol{G}_2)|^2 \le C'|\boldsymbol{G}_1-\boldsymbol{G}_2|^2$ a.e. in $\mathbb{R}^+\times(0,\mathcal{T})\times\Omega$. We deduce

$$|\boldsymbol{\tau}(t,\boldsymbol{x})|^2 \le \frac{C'\,\omega^2}{\mathfrak{We}^2}\int_0^{+\infty} m(s)\,|\boldsymbol{G}(s,t,\boldsymbol{x})|^2\,\mathrm{d}s.$$

Integrating with respect to $\boldsymbol{x}\in\Omega$ we obtain

$$\|\boldsymbol{\tau}\|^2_{L^2(\Omega)} \le \frac{C'\,\omega^2}{\mathfrak{We}^2}Y(t), \tag{5.3}$$

where we introduced $Y(t) = \displaystyle\int_0^{+\infty} m(s)\,\|\boldsymbol{G}(s,t,\cdot)\|^2_{L^2(\Omega)}\,\mathrm{d}s$.

Deformation gradient estimate – Taking the scalar product of the last equation of System (5.1) by $\boldsymbol{G}$ in $L^2(\Omega)$, we obtain

$$\begin{aligned}\frac{1}{2}\partial_t\big(\|\boldsymbol{G}\|^2_{L^2(\Omega)}\big) + \frac{1}{2\mathfrak{We}}\partial_s\big(\|\boldsymbol{G}\|^2_{L^2(\Omega)}\big) &= \int_\Omega (\boldsymbol{G}_1\cdot\nabla\boldsymbol{u})\cdot\boldsymbol{G}\\ &\quad + \int_\Omega(\boldsymbol{G}\cdot\nabla\boldsymbol{u}_2)\cdot\boldsymbol{G} - \int_\Omega(\boldsymbol{u}\cdot\nabla\boldsymbol{G}_2)\cdot\boldsymbol{G}.\end{aligned}$$

Using the Hölder inequality, we have the estimate

$$\begin{aligned}\frac{1}{2}\partial_t\big(\|\boldsymbol{G}\|^2_{L^2(\Omega)}\big) + \frac{1}{2\mathfrak{We}}\partial_s\big(\|\boldsymbol{G}\|^2_{L^2(\Omega)}\big) \le{}& \|\boldsymbol{G}_1\|_{L^\infty(\Omega)}\|\nabla\boldsymbol{u}\|_{L^2(\Omega)}\|\boldsymbol{G}\|_{L^2(\Omega)}\\ &+ \|\nabla\boldsymbol{u}_2\|_{L^\infty(\Omega)}\|\boldsymbol{G}\|^2_{L^2(\Omega)}\\ &+ \|\boldsymbol{u}\|_{L^{\frac{2d}{d-2}}(\Omega)}\|\nabla\boldsymbol{G}_2\|_{L^d(\Omega)}\|\boldsymbol{G}\|_{L^2(\Omega)}.\end{aligned}$$

Due to the Sobolev continuous injection $H^1(\Omega)\hookrightarrow L^{\frac{2d}{d-2}}(\Omega)$, the Poincaré inequality and the Young inequality, we obtain for all $\varepsilon>0$:

$$\partial_t\big(\|\boldsymbol{G}\|^2_{L^2(\Omega)}\big) + \frac{1}{\mathfrak{We}}\partial_s\big(\|\boldsymbol{G}\|^2_{L^2(\Omega)}\big) \le \varepsilon\|\nabla\boldsymbol{u}\|^2_{L^2(\Omega)} + C_Y(t)\|\boldsymbol{G}\|^2_{L^2(\Omega)}, \tag{5.4}$$

where the function

$$C_Y(t) = \sup_{s\in\mathbb{R}^+}\left\{\frac{2}{\varepsilon}\|\boldsymbol{G}_1\|^2_{L^\infty(\Omega)} + \|\nabla\boldsymbol{u}_2\|_{L^\infty(\Omega)} + \frac{2\,C^2}{\varepsilon}\,\|\nabla\boldsymbol{G}_2\|^2_{L^d(\Omega)}\right\} \in L^1(0,\mathcal{T}),$$

and where the constant C depends on Ω, p and d. Multiplying this estimate (5.4) by $m(s)$ and integrating for $s \in (0,+\infty)$ we obtain

$$Y'(t) + \frac{1}{\mathfrak{We}} \underbrace{\int_0^{+\infty} m(s)\partial_s\big(\|\boldsymbol{G}\|^2_{L^2(\Omega)}\big)\,\mathrm{d}s}_{\mathscr{I}} \leq \varepsilon\,\|\nabla \boldsymbol{u}\|^2_{L^2(\Omega)} + C_Y(t)Y(t). \quad (5.5)$$

Using a integration by part, the integral $\mathscr{I}$ becomes

$$\mathscr{I} = \int_0^{+\infty} -m'(s)\|\boldsymbol{G}\|^2_{L^2(\Omega)}\,\mathrm{d}s + \Big[m(s)\|\boldsymbol{G}\|^2_{L^2(\Omega)}\Big]_0^{+\infty}.$$

Using the following arguments:
- The memory function m is non-increasing, that is $-m' \geq 0$;
- The function $\boldsymbol{G}$ is bounded on $\mathbb{R}^+ \times (0,\mathcal{T}) \times \Omega$ and $m \in L^1(\mathbb{R}^+)$, that is $\lim\limits_{s\to+\infty} m(s)\|\boldsymbol{G}\|^2_{L^2(\Omega)} = 0$;
- We have the following development with respect to the variable s for $\boldsymbol{G}$:

$$\boldsymbol{G}(s) = \boldsymbol{G}\big|_{s=0} + s\,\partial_s\boldsymbol{G}\big|_{s=0} + o(s) \sim s\,\mathfrak{We}\,\nabla\boldsymbol{u}.$$

Moreover, $m \in L^1(\mathbb{R}^+)$ so that $\lim\limits_{s\to 0} m(s)\|\boldsymbol{G}\|^2_{L^2(\Omega)} = 0$.

Hence the integral $\mathscr{I}$ is non-negative so that the estimate (5.5) now reads

$$Y'(t) \leq \varepsilon\,\|\nabla\boldsymbol{u}\|^2_{L^2(\Omega)} + C_Y(t)Y(t). \quad (5.6)$$

Uniqueness result – Finally, adding (5.2) and (5.6) with the choice $\varepsilon = 1-\omega$, and using the estimate (5.3), we obtain

$$(Y+Z)'(t) \leq C_{YZ}(t)\,(Y+Z)(t),$$

where the function C_{YZ} is a linear combination of C_Y and C_Z. In particular we have $C_{YZ} \in L^1(0,\mathcal{T})$. The Gronwall lemma and the initial condition $Y(0) = Z(0) = 0$ imply that $Y = Z = 0$. We deduce that $\boldsymbol{u} = \boldsymbol{0}$ and $\boldsymbol{G} = \boldsymbol{0}$ and that consequently the stress $\boldsymbol{\tau} = \boldsymbol{0}$ and the pressure p is constant in $(0,\mathcal{T}) \times \Omega$. □

6. Proof for the global existence with small data

This section is devoted to the proof of the global existence theorem 3.3. Arguing as in the proof of Theorem 3.1, we introduce the space $\mathscr{B}(\mathcal{T})$, the subspaces $\mathscr{H}(\mathcal{T},R_1,R_2,R_3)$ and the mapping Φ.
For $(\overline{\boldsymbol{u}},\overline{\boldsymbol{G}},\overline{\boldsymbol{\tau}}) \in \mathscr{H}(\mathcal{T},R_1,R_2,R_3)$ and $(\boldsymbol{u},\boldsymbol{G},\boldsymbol{\tau}) = \Phi(\overline{\boldsymbol{u}},\overline{\boldsymbol{G}},\overline{\boldsymbol{\tau}})$ recall that we have the following estimates (see the estimates (4.18), (4.20) and (4.21)):

$$\begin{aligned}\|\boldsymbol{u}\|_{L^r(0,\mathcal{T};W^{2,p}(\Omega))} + \|\partial_t\boldsymbol{u}\|_{L^r(0,\mathcal{T};L^p(\Omega))} \leq \frac{C_1}{1-\omega}\Big(\|\boldsymbol{u}_0\|_{D^r_p(\Omega)} \\ + \|\boldsymbol{f}\|_{L^r(0,\mathcal{T};L^p(\Omega))} + \mathfrak{Re}\,C\,\mathcal{T}^\alpha R_1^\beta\|\boldsymbol{u}_0\|^\gamma_{D^r_p(\Omega)} + \mathfrak{Re}\,C\,\mathcal{T}^\delta R_1^2 + \mathcal{T}^{\frac{1}{r}}R_3\Big),\end{aligned} \quad (6.1)$$

$$\begin{aligned}\|\boldsymbol{G}\|_{L^\infty(\mathbb{R}^+\times(0,\mathcal{T});W^{1,p}(\Omega))} + \|\partial_t\boldsymbol{G}\|_{L^\infty(\mathbb{R}^+;L^r(0,\mathcal{T};L^p(\Omega)))} \\ \leq C_2\big(1+R_1\big)\exp\big(C_3\mathcal{T}^{1-\frac{1}{r}}R_1\big),\end{aligned} \quad (6.2)$$

$$\|\boldsymbol{\tau}\|_{L^\infty(0,\mathcal{T};W^{1,p}(\Omega))} + \|\partial_t\boldsymbol{\tau}\|_{L^r(0,\mathcal{T};L^p(\Omega))} \leq \frac{\omega}{\mathfrak{We}}F_0(R_2). \tag{6.3}$$

Note that the constants C_1, C_2, C_3, C and the function F_0 introduced in these three estimates do not depend on ω. For a time $\mathcal{T} > 0$ given, we successively choose

$$R_1^\star = \frac{1-\omega}{2C_1\,\mathfrak{Re}\,C\,\mathcal{T}^\delta},$$
$$R_2^\star = C_2\big(1+R_1^\star\big)\exp\big(C_3\mathcal{T}^{1-\frac{1}{r}}R_1^\star\big) + \|\boldsymbol{G}_{\mathrm{old}}\|_{L^\infty(\mathbb{R}^+;W^{1,p}(\Omega))},$$
$$R_3^\star = \frac{\omega}{\mathfrak{We}}F_0(R_2^\star).$$

We verify that, for $\boldsymbol{u}_0$, $\boldsymbol{f}$ small enough in their norms, and for ω small enough too, $\mathscr{H}(\mathcal{T},R_1^\star,R_2^\star,R_3^\star) \neq \emptyset$ (that is (4.12) holds). For such choices of $R_1^\star$, $R_2^\star$, ω and small norms of $\boldsymbol{u}_0$, $\boldsymbol{f}$ and $\boldsymbol{G}_{\mathrm{old}}$, the inequalities (6.1), (6.2) and (6.3) imply that $\Phi(\mathscr{H}(\mathcal{T},R_1^\star,R_2^\star,R_3^\star)) \subset \mathscr{H}(\mathcal{T},R_1^\star,R_2^\star,R_3^\star)$. Moreover the function Φ is continuous and $\mathscr{H}(\mathcal{T},R_1^\star,R_2^\star,R_3^\star)$ is a convex compact subset of $\mathscr{B}(\mathcal{T})$. We conclude the proof using the Schauder's theorem again. □

7. Conclusion

In this article we are interested in the mathematical properties of models for viscoelastic flows. We have shown that many known results for differential laws could be adapted to integral models. Nevertheless some differences persist and we present here some possible viewpoints:

– Our results are formulated in the $L^r - L^p$ context, following [15]. It seems possible to reformulate them for more regular solutions in the H^s context, following [21].

– The result concerning the global existence with small data (Theorem 3.3) is proved for the relaxation parameter ω small enough only, that is to say for the flows which are not too elastic. For the differential models, this assumption can be removed, see for instance [10, 33]. In this case, the results on differential models strongly use the structure of the equation, and it seems difficult to adapt such methods for integral models (see also Remark 3.4).

– There exist differential models which have no apparent equivalent in terms of integral models, for instance the co-rotational Oldroyd model. This study does not cover these cases (but they fall within work of C. Guillopé and J.-C. Saut [21, 22]). Similarly, there are also integral models more general than those studied here. In [42], R. I. Tanner introduce models where the memory m also depends on the invariants I_1 and I_2. It might be interesting to study these models from a theoretical point of view and to observe whether the approach taken here can be adapted.

– On the other hand, it is possible that classical integral models perform better than differential models of Oldroyd type. In fact, most of these models have a stress which is naturally bounded *via* the definition of $\mathcal{S}$, see the examples given by Equations (2.22), (2.23) or (2.24), and Appendix A. While obtaining a weak solution seems very difficult, knowing *a priori* a bound on the stress is an interesting information (see [9] for an example of a criterium for the explosion in the Oldroyd model).

– We note that we are not referring to the case of steady flows, the reason being that the case is largely dealt with by Renardy in [36]. We also mention the doctoral study by M. H. Sy [41] about steady flows, as well as one-dimensional problems for integral models.

– Finally, the theoretical results shown in this paper allow us to consider several possible lines of study on models of integral type. The well-known results for the differential model can be generalized to integral models. For instance, the one dimensional shearing motions and Poiseuille flows admit global existence for usual differential models, see [19]. In this regard, the work of A.C.T. Aarts and A.A.F. van de Ven [1] are interesting: they study the Poiseuille flow of a K-BKZ model. Would it be possible to prove global existence for such one dimensional flows when we use more general integral models ? Another possible generalization concerns the behavior of viscoelastic flows in thin geometries (in the fields such as polymer extrusion or lubrication), or in thin free-surface flows (to study mudslide or oil slick). Recent work [2, 3] and [7] can provide answers to the differential models, and we can imagine the same kind of work for integral models.

Appendix A. Tensors and the strain measure function $\mathcal{S}$

A.1. Some remarks on the invariants of the Finger tensor ($d = 3$)

For a matrix $\boldsymbol{B} \in \mathcal{L}(\mathbb{R}^3)$, we usually define three invariants:

$$I_1 = \mathrm{Tr}(\boldsymbol{B}), \quad I_2 = \frac{1}{2}\big((\mathrm{Tr}(\boldsymbol{B}))^2 - \mathrm{Tr}(\boldsymbol{B}^2)\big), \quad I_3 = \det \boldsymbol{B}.$$

We specify in this subsection some properties of these invariants in the context studied here, that is when $\boldsymbol{B}$ represents a Finger tensor of an incompressible flow.

First of all, this incompressibility condition implies that $\det \boldsymbol{B} = 1$. Consequently the third invariant I_3 is useless. Next, using the Cayley-Hamilton theorem we have $\boldsymbol{B}^{-1} = \boldsymbol{B}^2 - I_1 \boldsymbol{B} + I_2\, \boldsymbol{\delta}$ and we deduce that

$$I_2 = \mathrm{Tr}(\boldsymbol{B}^{-1}).$$

By definition, $\boldsymbol{B} = {}^T\boldsymbol{F} \cdot \boldsymbol{F}$ is real positive-definite matrix, and consequently it is diagonalizable. Using a basis formed by its eigenvectors, we have $I_1 = \lambda_1 + \lambda_2 + \lambda_3$ whereas $\lambda_1\lambda_2\lambda_3 = 1$. An inequality of arithmetic and geometric means indicates that $I_1 \geq 3$, and in a similar way we prove that $I_2 \geq 3$. We deduce that for all $\beta \in [0, 1]$ we have

$$\beta I_1 + (1 - \beta) I_2 \geq 3.$$

That mathematically justifies the PSM model (2.22) and the Wagner model given by (2.23).

A.2. Notion of derivative for the Strain measure tensor

The strain measure function $\mathcal{S} : \mathcal{L}(\mathbb{R}^d) \longrightarrow \mathcal{L}(\mathbb{R}^d)$ can be viewed as an application $\mathbb{R}^{d\times d} \longrightarrow \mathbb{R}^{d\times d}$. For $1 \le i, j \le d$, we introduce the matrix

$$\boldsymbol{E}_{ij} = \begin{array}{cc} \left(\begin{array}{ccc} 0 & & 0 \\ & 1 & \cdots \\ 0 & \vdots & 0 \end{array}\right) & \begin{array}{c} \\ j \\ \\ \end{array} \\ \begin{array}{ccc} & i & \end{array} & \end{array}$$

as element of the basis of $\mathcal{L}(\mathbb{R}^d)$ and we define the differential of $\mathcal{S}$ from its Jacobian $\mathcal{S}' := \left(\dfrac{\partial \mathcal{S}_{k\ell}}{\partial \boldsymbol{E}_{ij}}\right)_{ijk\ell}$. Note that in this paper, we use the notation $\partial_{(ij)}\mathcal{S}_{k\ell} := \dfrac{\partial \mathcal{S}_{k\ell}}{\partial \boldsymbol{E}_{ij}}$.

A.3. Norm for the 4-tensor

The notion of derivative introduced above involves the use of tensors of order 4. Recall that for a 2-tensor $\boldsymbol{A} = (\boldsymbol{A})_{ij}$, we use the usual algebra norm defined by $|\boldsymbol{G}|^2 := \mathrm{Tr}({}^T\boldsymbol{G} \cdot \boldsymbol{G})$. For a 4-tensors $\boldsymbol{H} = (\boldsymbol{H})_{ijk\ell}$, we introduce the following algebra norm:

$$|\boldsymbol{H}|^2 := \sum_{i,j,k,\ell} \boldsymbol{H}_{ijk\ell}^2.$$

We will note that this norm having the following property: $|\boldsymbol{A} \otimes \boldsymbol{B}| = |\boldsymbol{A}|\,|\boldsymbol{B}|$ for any 2-tensors $\boldsymbol{A}$ and $\boldsymbol{B}$.

A.4. Example for a PSM model

Consider the example function corresponding to a PSM model (see Equation (2.22) with $\alpha = 4$ and $\beta = 1$):

$$\mathcal{S} : \boldsymbol{G} \in \mathcal{L}(\mathbb{R}^d) \longmapsto \frac{\boldsymbol{B}}{1 + \mathrm{Tr}(\boldsymbol{B})} \in \mathcal{L}(\mathbb{R}^d) \qquad \text{where} \quad \boldsymbol{B} = {}^T\boldsymbol{G} \cdot \boldsymbol{G}.$$

Proposition A.1. *For all $\boldsymbol{G} \in \mathcal{L}(\mathbb{R}^d)$ we have $|\mathcal{S}(\boldsymbol{G})| \le 1$.*

Proof. Using the norm on the 2-tensor, we have for all $\boldsymbol{G} \in \mathcal{L}(\mathbb{R}^d)$:

$$|\mathcal{S}(\boldsymbol{G})| = \frac{|{}^T\boldsymbol{G} \cdot \boldsymbol{G}|}{1 + |\boldsymbol{G}|^2} \le \frac{|\boldsymbol{G}|^2}{1 + |\boldsymbol{G}|^2},$$

that implies that $\mathcal{S}$ is bounded by the constant 1. □

Proposition A.2. *For all $\boldsymbol{G} \in \mathcal{L}(\mathbb{R}^d)$ we have $|\mathcal{S}'(\boldsymbol{G})| \le \dfrac{2(1+\sqrt{d})}{|\boldsymbol{G}|}$.*

Proof. The derivative of the function $\mathcal{S}$ is a function with values in the set of 4-tensors:

$$\begin{aligned}\mathcal{S}'(\boldsymbol{G})_{ijk\ell} &= \partial_{(ij)}\Big(\frac{\boldsymbol{B}_{k\ell}}{1+\mathrm{Tr}(\boldsymbol{B})}\Big)\\ &= \frac{\partial_{(ij)}(\boldsymbol{B}_{k\ell})}{1+\mathrm{Tr}(\boldsymbol{B})} - \frac{\partial_{(ij)}(\mathrm{Tr}(\boldsymbol{B}))\boldsymbol{B}_{kl}}{(1+\mathrm{Tr}(\boldsymbol{B}))^2}\\ &= \frac{(\boldsymbol{\delta}_{kj}\boldsymbol{G}_{i\ell}+\boldsymbol{\delta}_{\ell j}\boldsymbol{G}_{ik})}{1+\mathrm{Tr}(\boldsymbol{B})} - \frac{2\boldsymbol{G}_{ij}\boldsymbol{B}_{kl}}{(1+\mathrm{Tr}(\boldsymbol{B}))^2},\end{aligned}$$

Taking the 4-tensor norm, we deduce that

$$|\mathcal{S}'(\boldsymbol{G})| \le \frac{2|\boldsymbol{\delta}|\,|\boldsymbol{G}|}{1+|\boldsymbol{G}|^2} + \frac{2|\boldsymbol{G}|^3}{(1+|\boldsymbol{G}|^2)^2}$$

Since $|\boldsymbol{\delta}| = \sqrt{d}$, this implies that $|\boldsymbol{G}|\,|\mathcal{S}'(\boldsymbol{G})|$ is bounded by $2(1+\sqrt{d})$. □

Appendix B. Gronwall type lemma

Lemma B.1. *Let $f : \mathbb{R}^+ \mapsto \mathbb{R}^+$ a positive and locally integrable function. If a function $y : \mathbb{R}^+\times\mathbb{R}^+ \mapsto \mathbb{R}$ satisfies, for all $(s,t)\in\mathbb{R}^+\times\mathbb{R}^+$:*

$$\partial_t y(s,t) + \frac{1}{\mathfrak{We}}\partial_s y(s,t) \le f(t)\,y(s,t) \tag{B.1}$$

then we have, for all $(s,t)\in\mathbb{R}^+\times\mathbb{R}^+$:

$$y(s,t) \le \zeta(s,t)\exp\Big(\int_0^t f(t')\,\mathrm{d}t'\Big), \tag{B.2}$$

where $\zeta(s,t) = \begin{cases} y\big(s-\dfrac{t}{\mathfrak{We}},0\big) & \text{if } t\le \mathfrak{We}\,s,\\ y(0,t-\mathfrak{We}\,s) & \text{if } t> \mathfrak{We}\,s.\end{cases}$

Proof. Introducing the new variables $u=\frac12(\mathfrak{We}\,s+t)$ and $v=\frac12(\mathfrak{We}\,s-t)$, we can write the first equation of System (B.1) as a system on the function $z(u,v)=y(s,t)$:

$$\partial_u z(u,v) \le f(u-v)\,z(u,v).$$

Since the function f is locally integrable, we obtain

$$\partial_u\Big[z(u,v)\exp\Big(-\int_0^{u-v} f(t')\,\mathrm{d}t'\Big)\Big] \le 0.$$

Integrating this relation between $|v|$ and u, we deduce

$$z(u,v)\exp\Big(-\int_0^{u-v} f(t')\,\mathrm{d}t'\Big) \le z(|v|,v)\exp\Big(-\int_0^{|v|-v} f(t')\,\mathrm{d}t'\Big).$$

Due to the positivity of the function f, the exponential term in the last equation being less than 1. According to the sign of v, we have $z(|v|,v) = y(0,t-\mathfrak{We}\,s)$ or $z(|v|,v) = y(s-\frac{t}{\mathfrak{We}},0)$. That implies the result (B.2) announced in the lemma. □

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Laurent Chupin
Laboratoire de Mathmatiques, UMR CNRS 6620
Université Blaise Pascal, Campus des Cézeaux
63177 Aubière Cedex, France
e-mail: `laurent.chupin@math.univ-bpclermont.fr`